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Benjamin-Ono periodic bifurcating water waves in presence of an essential spectrum
Bifurcation en présence d'un spectre essentiel de vagues périodiques progressives de Benjamin-Ono

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Abstract

The mathematical study of travelling waves in the potential flow of two superposed layers of perfect fluid can be set as an ill-posed evolutionary problem, in which the horizontal unbounded space variable plays the role of "time". In this paper we consider two problems for which the bottom layer of fluid is infinitely deep: for the first problem, the upper layer is bounded by a rigid top and there is no surface tension at the interface; for the second problem, there is a free surface with a large enough surface tension. In both problems, the linearized operator L_ε (where ε is a combination of the physical parameters) around 0 possesses an *essential spectrum filling the entire real line*, with in addition a simple eigenvalue in 0. Moreover, for $\varepsilon < 0$, there is a pair of imaginary eigenvalues which meet in 0 when $\varepsilon = 0$ and which disappear in the essential spectrum for $\varepsilon > 0$. For $\varepsilon > 0$ small enough, we prove in this paper the existence of a two parameter family of periodic travelling waves (corresponding to periodic solutions of the dynamical system). These solutions are obtained in showing that the full system can be seen as a perturbation of the Benjamin-Ono equation. The periods of these solutions run on an interval (T_0, ∞) possibly except a discrete set of isolated points.

Keywords: nonlinear water waves, travelling waves, infinite-dimensional reversible dynamical system, essential spectrum, periodic orbits.
AMS classification: 35B32, 76B15

Résumé

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La recherche d'ondes progressives dans un système de deux couches superposées de fluides parfaits peut s'écrire comme un problème d'évolution mal posé, pour lequel la variable horizontale non bornée remplace le temps. Dans cet article, on étudie deux problèmes pour lesquels la couche inférieur de fluide est de profondeur infinie : dans le premier problème la couche supérieure est bornée par une surface rigide et il n'y a pas de tension de surface à l'interface; pour le deuxième problème, la surface est libre mais avec une tension de surface élevée. Pour les deux problèmes, l'opérateur linéarisé à l'origine L_ε (où ε est une combinaison des paramètres physiques) possède, en plus d'une valeur propre simple en 0, *un spectre essentiel sur tout l'axe réel*. De plus, pour $\varepsilon < 0$, il y a une paire de valeurs propres imaginaires pures, qui se rencontrent à l'origine pour $\varepsilon = 0$, et qui disparaissent dans le spectre essentiel pour $\varepsilon > 0$. Pour $\varepsilon > 0$ assez petit, on montre l'existence d'une famille à deux paramètres d'ondes progressives périodiques (qui correspondent à des solutions périodiques du système dynamique). Ces solutions sont obtenues en montrant que le système dynamique peut se réduire à une perturbation de l'équation de Benjamin-Ono. Les périodes de ces solutions appartiennent à un intervalle (T_0, ∞) à l'exception possible d'un ensemble discret de points isolés.

Mot clés : vagues non linéaires, ondes progressives, systèmes dynamiques réversibles de dimension infinie, spectre essentiel, orbites périodiques.
Classification AMS : 35B32, 76B15

1 Introduction

The search for travelling waves in a system of superposed perfect fluid layers, having a potential flow in each layer, may be formulated as a "spatial dynamical system". Such "spatial dynamics" was introduced in the 80's by K. Kirchgässner [8] for solving an elliptic problem in a strip. Writing the system in the frame moving with the velocity of the travelling wave, we look for steady solutions. In choosing the unbounded spatial coordinate "x" as the evolutionary variable (and then replacing "time") the problem reads

$$\frac{dU}{dx} = F(U), \ x \in \mathbb{R}, \tag{1}$$

where U takes its values in general in an infinite dimensional space. The initial value problem is then ill-posed, but looking for bounded solutions on the real line leads to a sort of "boundary value" problem. When the initial problem has a symmetry $x \to -x$, then the vector field F anti-commutes with a symmetry S, and the dynamical system is said to be "reversible". An easy consequence is that if $U(x)$ is a solution, then $SU(-x)$ is also a solution. The spatial dynamics consideration allows in particular to study the asymptotics at infinity. For instance, periodic solutions or homoclinic orbits of (1) correspond respectively to periodic travelling waves or to solitary waves. A review of results concerning problems where all layers have finite thickness and treated as a spatial dynamical system is made in the paper [5].

In the present paper, we study two problems for which the bottom layer of fluid is infinitely deep. The two problems consist in a system of two superposed layers of immiscible perfect fluids (densities ρ_1 (upper layer) and ρ_2 (bottom layer)) assuming that there is no surface tension at the interface and assuming that the flow is potential in each layer and subject to gravity. The thickness at rest of the upper layer is h while the bottom layer is infinitely deep. We are interested in travelling waves of horizontal velocity c. The dimensionless parameters are $\rho = \rho_1/\rho_2 \in (0,1)$ and $\lambda = gh/c^2$ (inverse of (Froude number)^2). In the first problem we assume that the upper layer is bounded above by a rigid horizontal top (see Figure 1). Such a case was treated, using a different formulation, by Amick [1] and Sun [11], where the existence of a solitary wave is shown, asymptotically looking like the Benjamin-Ono solitary wave.

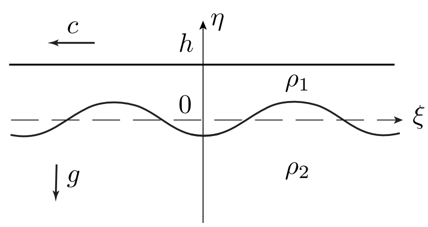

Figure 1: Problem 1. Two layers, the upper one being bounded by a rigid top.

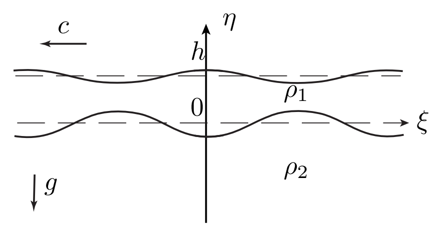

Figure 2: Problem 2. Two layers, free upper surface with large surface tension.

In the second problem we assume that there is a free upper surface with surface tension T (see Figure 2), and there is a new dimensionless number $b = T/(\rho_1 hc^2)$ (Weber number). Notice that this situation is studied in [7] in the case when $b = 0$, where the existence of generalized solitary waves is shown, the principal part being again solution of the Benjamin-Ono equation. In the present paper, the physical problem is different since we consider it for a large enough parameter b.

We show in section 2 how these problems can be formulated as a *reversible dynamical system* (1), where $U = 0$ corresponds to a uniform state, and where F depends on the parameters λ and ρ (and b for problem 2) . The Galilean invariance of the physical problem induces a reflection symmetry of the system in the moving frame. This leads to the reversibility of system (1).

The study near the origin of dynamical systems (1), which depend on a parameter ε consists first on the computation of the spectrum of the linearized operator

L_ε around an equilibrium, taken at the origin. The system (1) reads

$$\frac{dU}{dx} = L_\varepsilon U + N_\varepsilon(U),\ x \in \mathbb{R}, \tag{2}$$

where N_ε is the non linear term. The reversibility symmetry S leads to the symmetry of the eigenvalues with respect to real and imaginary axis. In the case when the linear operator L_ε has a "spectral gap", the system (2) can be reduced to a system of finite dimensional ordinary differential equations by using the center manifold reduction theorem. This leads to the study of a perturbed reversible normal form. In such a case the description near the imaginary axis of the spectrum of the linear operator is sufficient to understand the dynamics of small reversible solutions of (2) (see for instance [5]).

Figure 3: Spectrum of L_ε

In the present work, the linearized operator around 0: $L_\varepsilon = D_U F(0)$, where $\varepsilon = \rho - \lambda(1-\rho)$ for problem 1, and $\varepsilon = 1 - \lambda(1-\rho)$ for problem 2, possesses an *essential spectrum on the entire real axis*. Therefore, there is no spectral gap and the center manifold reduction cannot apply. In addition to the essential spectrum, the linear operator L_ε possesses a simple eigenvalue in 0. This eigenvalue results from the existence of a one parameter family of stationary solutions of (1) which correspond physically to the sliding with uniform velocity of the upper surface over the bottom one. More precisely, the family of equilibria of (1) reads $U(x) = u\xi_0$, $u \in \mathbb{R}$, where ξ_0 is the (symmetric) eigenvector associated with the 0 eigenvalue. Moreover (when b is large enough for problem 2), for $\varepsilon < 0$, L_ε has two conjugated imaginary eigenvalues, which meet at the origin for $\varepsilon = 0$ and which disappear in the essential spectrum for $\varepsilon > 0$ (see Figure 3).

This bifurcation has been encountered in [3], in which the existence of a one parameter family of bifurcated homoclinic solutions of (1) approximated by the Benjamin-Ono solitary wave is proved. Because of the presence of the essential spectrum in this paper, the sole description of the spectrum is not sufficient to prove the existence of these solutions. In particular, the proof of the existence of these solutions is based on a delicate study of the Fourier transform of (2), then the main tool is the resolvent operator $(ik - L_\varepsilon)^{-1}$ for k real. In presence of an essential spectrum on the real axis, the singularity of the resolvent in $k = 0$ is unknown. That's why a meticulous description of the resolvent is performed in [3] : some assumptions on the resolvent of L_ε are given in order to describe the singularity in $k = 0$. It is checked in [3] that these assumptions are satisfied in both water-wave problems presented above. Therefore, there exists a one parameter family of solitary waves for problems 1 and 2 (corresponding to the homoclinic solutions in the spatial dynamics formulation). These waves are approximated by the Benjamin-Ono solitary waves (see [4], [10]).

The aim of the present paper is to prove the existence of a family of bifurcating periodic solutions of (2). The bifurcation of periodic solutions might be studied for $\varepsilon < 0$ in using the way of [6] which generalizes the Lyapunov-Devaney method for finite dimensional reversible vector fields. However, there is an additional technical difficulty due to the fact that the two imaginary eigenvalues are close to 0 together with the occurence of 0 in the continuous spectrum as in [6]. Notice that the periodic waves obtained in [7] are of the same nature as in [6]. On the contrary, the solutions which interest us below are the bifurcating periodic solutions for $\varepsilon > 0$. These ones are of different nature, and similar to the ones observed in the case of a three-dimensional one parameter family of reversible vector fields where 0 is a fixed point and where in addition to the 0 fixed eigenvalue of the linearizeed operator, a pair of imaginary eigenvalues collide at 0 and become a pair of two real opposite eigenvalues. In this three-dimensional case, for every homoclinic orbit near 0, there is a one-parameter family of periodic orbits starting from a point at the elliptic equilibrium and growing until the homoclinic to the hyperbolic equilibrium (see for example the result in [5] for the corresponding situation). We show below that the periodic solutions found in the present paper are approximated by solutions of the periodic version of the Benjamin-Ono equation. These periodic solutions correspond to periodic travelling waves in both water-wave problems. Actually, we can prove the existence of such periodic solutions for the general dynamical system having the spectrum as in Figure 3, and satisfying the generic properties presented for the water-wave problems in sections 2.3 and 2.4. Therefore, the properties on the resolvent and on the non linear term (see section 2.3 and 2.4) for the water-wave problems are written in the general frame of spatial dynamics. These properties correspond to weaker assumptions than the ones made in [3].

The idea for the proof of the existence of periodic solutions is to reduce the full system to a non local scalar equation. This equation turns out to be a perturbation of the Benjamin-Ono equation which reads as follows

$$u + \frac{2\pi}{T}\mathcal{H}_\sharp(u') + ac_0u^2 = c, \tag{3}$$

where a and c_0 are two coefficients given below (see (6) and (7)) and they are related to the physical parameters of problem 1 and 2, aT is the period of the solutions and c is an arbitrary constant. The operator $\mathcal{H}_\sharp$ is the Hilbert transform for the 2π-periodic functions defined by the relation

$$\mathcal{H}_\sharp(f)(s) = \frac{1}{2\pi}\text{p.v.}\int_{-\pi}^{\pi} \frac{f(\tau)}{\tan(\frac{1}{2}(s-\tau))}d\tau.$$

The Hilbert transform can also be defined by the following relations $\mathcal{H}_\sharp(\cos) = \sin$, $\mathcal{H}_\sharp(\sin) = -\cos$ and $\mathcal{H}_\sharp(const.) = 0$. We know (see [2]) that the function v_p defined for $p > 2\pi$ by

$$v_p(s) = \frac{v_0}{\cos^2(s/2) + \left(\frac{pv_0}{2\pi}\right)^2 \sin^2(s/2)}, \tag{4}$$

where $v_0 = 1 - \sqrt{1 - (2\pi/p)^2}$ is a 2π-periodic solution (unique up to translation) of (3) when $T = p$, $ac_0 = -1$ and $c = 0$, and we can obtain explicitly other periodic solutions for $c \neq 0$.

Before presenting the results of this paper, we need to perform a scaling in (2) which dilates the spectrum of L_ε of a factor $1/\varepsilon$ (see section 2.2 for the explicit scaling). The new system reads

$$\frac{dU}{dx} = \mathcal{L}_\varepsilon U + \mathcal{N}_\varepsilon(U), \; U(x) \in \mathbb{D} \subset \mathbb{H}. \tag{5}$$

The next step is the study of the resolvent operator $(ik - \mathcal{L}_\varepsilon)^{-1}$ on the imaginary axis (i.e. for k real) near the origin and when k is large (see section 2.3). In particular, the dispersion equation of both physical problems can be written as $ik\varepsilon\Delta(\varepsilon, k) = 0$, where for k real

$$\Delta(\varepsilon, k) = 1 + a|k| + O(\varepsilon k^2), \tag{6}$$

and where $a = 1$ for problem 1, and $a = \lambda(\lambda - 1)^{-1}$ for problem 2. The solutions of the dispersion equation give the eigenvalues $\sigma = ik$ of the linearized operator $\mathcal{L}_\varepsilon$. Some needed properties on the non linear term $\mathcal{N}_\varepsilon$ are also given in section 2.4.

With the properties on the resolvent and on the non linear term, presented in section 2, we obtain in section 3 the existence of a two parameter family of periodic solutions of (5), corresponding to periodic travelling waves for the water-wave problems. This family is constructed in two steps. First we prove the existence of periodic solutions close to the equilibria $u\xi_0$ with $u \sim -1/(ac_0)$ (where $2c_0 = 3\rho$ for problem 1 and $2c_0 = -3\lambda/(\lambda-1)$ for problem 2) and with a period close to $2\pi a$ (recall that ξ_0 is the (symmetric) eigenvector associated with the 0 eigenvalue). Then we obtain the existence of periodic solutions with a period close to ap for almost all values of $p > 2\pi$.

Result of Theorem 3.2. *There exist $\varepsilon_0 > 0$, $\kappa_1 > \kappa_0 > 0$ and $A_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, equation (5) has a family of reversible periodic solutions $\mathcal{U}^A_{\kappa,\varepsilon}$, parametrized by A ($|A| < A_0$) and $\kappa \in [\kappa_0, \kappa_1]$. The period of these solutions is aT where T is given by*

$$T(\varepsilon, \kappa, A^2) = \frac{2\pi}{\kappa(1-\mu)}, \; \mu = \frac{1}{2}(\kappa^{-1}ac_0)^2 A^2 + O(\varepsilon),$$

where

$$c_0 = 3\rho/2 \text{ for problem 1 and } c_0 = -3\lambda/2(\lambda - 1) \text{ for problem 2.} \tag{7}$$

These solutions read

$$\mathcal{U}^A_{\kappa,\varepsilon}(x) = u^0_{\kappa,\varepsilon}\xi_0 + A\,\mathcal{U}_{\kappa,\varepsilon}(x) + O(A^2), \tag{8}$$

where ξ_0 *is the (symmetric) eigenvector associated with the 0 eigenvalue and*

$$u^0_{\kappa,\varepsilon} = -\frac{\kappa+1}{2ac_0} + O(\varepsilon),\ \textit{is a constant}, \tag{9}$$

and $\mathcal{U}_{\kappa,\varepsilon}(x) = \cos(2\pi x/(aT))\xi_0 + O(\varepsilon)$.

This theorem shows that the equilibria $U = u^0_{\kappa,\varepsilon}\xi_0$ are limits of periodic solutions of amplitude $O(A)$ tending towards 0. We can also prove the existence of a second family of equilibria which read $U = u\xi_0$ with $u \sim (\kappa-1)/(2ac_0)$, hence close to 0 when κ is close to 1. These last equilibria are the one considered in [3] to which the homoclinic solutions are connected.

In the previous Theorem, we proved the existence of ap/κ-periodic solutions where $p > 2\pi$ is close to 2π. In next Theorem, we prove that we can extend this result to "large periodic solutions" of period ap/κ, for almost all values of $p > 2\pi$.

Result of Theorem 3.3. *There exists a sub-set* $\mathcal{P}$ *of* $(2\pi, +\infty)$*, which differs from the interval* $(2\pi, +\infty)$ *by a discrete set without point of accumulation, for which the following result holds : for all compact set* $\mathcal{K} \subset \mathcal{P}$*, there exist* $\varepsilon_0 > 0$*,* $\kappa_1 > \kappa_0 > 0$ *such that equation (5) has a family of periodic solutions* $\mathcal{V}^p_{\kappa,\varepsilon}$*, parametrized by* $p \in \mathcal{K}$ *and* $\kappa \in (\kappa_0, \kappa_1)$*. The period of these solutions is* aT *where* $T = p/\kappa$*, and* $\mathcal{V}^p_{\kappa,\varepsilon}$ *satisfies*

$$\mathcal{V}^p_{\kappa,\varepsilon}(x) = \left(-\frac{\kappa}{ac_0}v_p(2\pi x/(aT)) + \frac{\kappa-1}{2ac_0}\right)\xi_0 + O(\varepsilon),$$

where v_p *is defined in (4).*

The periodic solutions constructed above correspond to periodic travelling waves for both problems as presented in Figure 1 and Figure 2. We can compute the expression of the interface $Z_{I,1}$ for problem 1. This expression reads (using $2ac_0 = 3\rho$ and $a = 1$), in the unscaled variables

$$Z_{I,1}(x) \sim \varepsilon\left(-\frac{2\kappa}{3\rho}v_p\left(2\pi\kappa\varepsilon x/p\right) + \frac{\kappa-1}{3\rho}\right), \tag{10}$$

where $\varepsilon = \rho - \lambda(1-\rho)$, $\kappa \in (\kappa_0, \kappa_1)$ and $p \in \mathcal{K}$. This expression is valid for all the values of p close to 2π and we can expand it with respect to the amplitude A (see the result of Theorem 3.2 above)

$$Z_{I,1}(x) \sim -\varepsilon\left(\frac{\kappa+1}{3\rho} + A\cos(2\pi\varepsilon x/T)\right),$$

where T is close to $2\pi/\kappa$.

For problem 2, we obtain the following expressions for the interface $Z_{I,2}$ and the free surface Z (using $2ac_0 = -3$ and $a = \lambda(\lambda-1)^{-1}$)

$$Z_{I,2}(x) \sim -\varepsilon\frac{\lambda-1}{\lambda}\left(\frac{2\kappa}{3}v_p\left(2\pi\kappa\varepsilon x/(ap)\right) - \frac{\kappa-1}{3}\right), \tag{11}$$

$$Z(x) \sim \frac{\varepsilon}{\lambda}\left(\frac{2\kappa}{3}v_p\left(2\pi\kappa\varepsilon x/(ap)\right) - \frac{\kappa-1}{3}\right). \tag{12}$$

Notice that we also obtain the expansion of these expressions when p is close to 2π as for problem 1.

Notice that the periodic solutions found here are different from the ones found in [6] and [7]. In these two articles, the periodic solutions result from a generalisation of the Liapounov-Devaney theorem in presence of the resonance due to the eigenvalue 0 which lies in the essential spectrum. This would correspond here to a study with $\varepsilon < 0$, where a pair of eigenvalues sit on the imaginary axis.

The proof of these Theorems consists in the reduction of (5) to a scalar non local equation, which is a perturbation on the Benjamin-Ono equation (3). We first fix a real number $T > 0$ and we look for aT-periodic solution of (5). With the scaling $U(x) = \underline{U}(s)$ where $s = aTx/(2\pi)$ we now look for 2π-periodic solutions of the new system

$$\frac{2\pi}{aT}\frac{d\underline{U}}{ds} = \mathcal{L}_\varepsilon\underline{U} + \mathcal{N}_\varepsilon(\underline{U}). \tag{13}$$

We decompose a solution of (13) as follows

$$\underline{U} = u\xi_0 + \varepsilon Y,$$

where u is a 2π-periodic scalar function and Y is in a complementary space of ξ_0. The reduction technics consists in proving that Y can be written as a function of u (see Theorem 4.2)

The next step of the reduction consists in finding the equation satisfied by u (see Theorem 4.3).

Equation for u. *Let $\underline{U} = u\xi_0 + \varepsilon Y$ be a 2π-periodic reversible solution of (13). Then u satisfies the following equation*

$$u + \frac{2\pi}{T}\mathcal{H}_\sharp(u') + ac_0u^2 = c + O(\varepsilon), \tag{14}$$

where c is a constant of integration, c_0 is defined by (7) and $\mathcal{H}_\sharp$ is the Hilbert transform for 2π-periodic functions.

The rest of the paper is devoted to the resolution of (14). We first change the parameters in (14) as follows $T = p/\kappa$ and $c = (\kappa^2-1)/(4ac_0)$ with $p > 2\pi$ and $\kappa > 0$. We now consider the equation

$$u + \kappa\frac{2\pi}{p}\mathcal{H}_\sharp(u') + ac_0u^2 = \frac{\kappa^2-1}{4ac_0} + O(\varepsilon). \tag{15}$$

The search of stationary solutions of (15) leads to the existence of two solutions: the first one being precisely $u^0_{\kappa,\varepsilon}$ given in (9). The second solution is $u = (\kappa - 1)/(2ac_0) + O(\varepsilon)$. This solution is close to 0 for κ close to 1 and corresponds to the equilibria considered in [3].

Since the function $u_{\kappa,p}$ defined by

$$u_{\kappa,p} = -\frac{\kappa}{ac_0} v_p + \frac{\kappa - 1}{2ac_0}$$

is a solution of (15) when the term $O(\varepsilon)$ is zero (see [2]), we search even solutions u of (15) as a perturbation of $u_{\kappa,p}$, i.e. $u = u_{\kappa,p} + w$. The equation satisfied by w reads

$$L_p w = N_{\varepsilon,\kappa,p}(w), \tag{16}$$

where L_p is defined by

$$L_p w = w + \frac{2\pi}{p} \mathcal{H}_\sharp(w') - 2v_p w,$$

where v_p is defined in (4) and $N_{\varepsilon,\kappa,p}(w) = -\kappa^{-1} ac_0 w^2 + O(\kappa^{-1}\varepsilon)$. Actually, L_p is the Benjamin-Ono operator linearized around the solution $u_{\kappa,p}$.

We use the implicit function theorem to find solutions of (16). First we study the case when $p \simeq 2\pi$. Since $v_{2\pi} = 1$ we have $L_{2\pi} w = -w + \mathcal{H}_\sharp(w')$, then $L_{2\pi}$ is not invertible and its kernel in a space of even functions is spanned by the function cos. The Lyapounov-Schmidt method leads to a solution $w = A\cos + O(A^2 + \varepsilon)$ of (16) where $p \simeq 2\pi$ is a function of the amplitude A and of ε. These solutions give the solutions of Theorem 3.2 thanks to the reduction theorem.

We finally study the equation (16) with $p > 2\pi$. The difficulty is to know whether L_p is invertible. We know that it's not invertible in a space including odd functions since $\partial_s v_p$ is in the kernel for all the values of $p > 2\pi$. But the study made for $p \simeq 2\pi$ shows that L_p is invertible for even functions when $p > 2\pi$ is close to 2π. The family L_p, $p > 2\pi$ being analytic with respect to p, we can prove that L_p is invertible for even functions for $p \in \mathcal{P}$ where the set $\mathcal{P}$ differs from the interval $(2\pi, +\infty)$ by a discrete set with no accumulation point.

Thanks to the implicit function theorem we can solve (16) with respect to w. This leads to the existence of a family of solutions of (14) which read $u = u_{\kappa,p} + O(\varepsilon)$, and then lead to the solutions $\mathcal{V}^p_{K,\varepsilon}$ of Theorem 3.3.

2 Spatial dynamic formulation

In this section we show how we can obtain a "spatial dynamics" formulation (see [8]) to describe the water wave problems. We then linearize the vector field around the origin (which is a stationary solution of the spatial dynamical system) and we study its resolvent operator on the imaginary axis. We finally give some properties on the non linear term.

2.1 Formulation of the water-wave problems

In the moving reference frame, denoting by ξ, η the physical coordinates, the complex potential in layer j is denoted $w_j(\xi+i\eta)$ and the complex velocity $w_j'(\xi+i\eta) = u_j - iv_j$. For formulating both problems as a dynamical system we proceed as in [6] and [7] and use the change of coordinates used by Levi-Civita : the new unknowns are $\alpha_j + i\beta_j$, $j = 1, 2$ which are analytic functions of $w_j = x_j + iy$ where x_j is the velocity potential in the layer j and y is the stream function and where

$$w_j'(\xi + i\eta) = e^{\beta_j - i\alpha_j}.$$

Notice that α_j is the slope of the streamline and e^{β_j} is the modulus of the velocity in the region j. The interface is then given by $y = 0$ and the upper surface by $y = 1$. The region of the flow is $-\infty < \underline{y} < 0$ for fluid 2 and $0 < y < 1$ for fluid 1. We choose as the basic $\underline{x}$ coordinate the one given by the bottom layer (x_2) (and we notice that $dx_1/dx_2 = e^{\beta_{10}-\beta_{20}}$ which introduces a factor in the Cauchy-Riemann equations of the upper layer).

With this choice of coordinates we formulate our problems as a dynamical system (see [6])

$$\frac{d\underline{U}}{d\underline{x}} = F(\underline{U}), \tag{17}$$

with the following unknown for problem 1

$$[\underline{U}(\underline{x})](y) = (\beta_{20}(\underline{x}), \alpha_1(\underline{x}, y), \beta_1(\underline{x}, y), \alpha_2(\underline{x}, \underline{y}), \beta_2(\underline{x}, \underline{y}))^t,$$

where β_{20} is the trace of β_2 at $y = 0$. The right hand side of (17) reads

$$F(\underline{U}) = \begin{cases} -\lambda(1-\rho)e^{3\beta_{20}}\sin\alpha_{10} - \rho\frac{\partial\alpha_1}{\partial y}_{|y=0} e^{3(\beta_{10}-\beta_{20})}, \\ \left.\begin{array}{l} \frac{\partial\beta_1}{\partial y}e^{\beta_{10}-\beta_{20}} \\ -\frac{\partial\alpha_1}{\partial y}e^{\beta_{10}-\beta_{20}} \end{array}\right\} y \in (0,1), \\ \left.\begin{array}{l} \frac{\partial\beta_2}{\partial y} \\ -\frac{\partial\alpha_2}{\partial \underline{y}} \end{array}\right\} \underline{y} \in (-\infty, 0). \end{cases} \tag{18}$$

Equation (17) is understood in $\mathbb{H}$ where

$$\mathbb{H} = \mathbb{R} \times \left\{\mathcal{C}^0(0,1)\right\}^2 \times \left\{\mathcal{C}_1^0(\mathbb{R}^-)\right\}^2,$$

and $\underline{U}(\underline{x})$ lies in $\mathbb{D}$ where

$$\begin{aligned} \mathbb{D} = \mathbb{R} &\times \left\{\mathcal{C}^1(0,1)\right\}^2 \times \left\{\mathcal{C}_1^1(\mathbb{R}^-)\right\}^2 \\ &\cap \left\{\alpha_1(0) = \alpha_2(0), \alpha_1(1) = 0, \beta_{20} = \beta_2(0)\right\}, \end{aligned}$$

where we define the following Banach spaces

$$\begin{aligned} \mathcal{C}_1^0(\mathbb{R}^-) &= \left\{f \in \mathcal{C}^0(\mathbb{R}^-); \sup_{y\in\mathbb{R}^-} |f(y)|(1+|y|) < \infty\right\}, \\ \mathcal{C}_1^1(\mathbb{R}^-) &= \left\{f \in \mathcal{C}_1^0(\mathbb{R}^-); f' \in \mathcal{C}_1^0(\mathbb{R}^-)\right\}. \end{aligned}$$

The norm in $\mathbb{H}$ for $V = (a, f_1, g_1, f_2, g_2)^t \in \mathbb{H}$ is defined by

$$\|V\|_{\mathbb{H}} = |a| + \|f_1\|_\infty + \|g_1\|_\infty + \|f_2\|_{1,\infty} + \|g_2\|_{1,\infty},$$

with

$$\|f\|_{1,\infty} = \sup_{y \in \mathbb{R}^-} |f(y)|(1 + |y|),$$

and we obtain the norm in $\mathbb{D}$ by adding the norms of f_i' and g_i'.

For problem 2, the unknown is defined by

$$[\underline{U}(\underline{x})](y) = (\beta_{20}(\underline{x}), Z(\underline{x}), \alpha_{11}(\underline{x}), \alpha_1(\underline{x}, y), \beta_1(\underline{x}, y), \alpha_2(\underline{x}, \underline{y}), \beta_2(\underline{x}, \underline{y}))^t,$$

where $1 + \frac{1}{2\lambda}(1 - e^{-2\lambda Z(\underline{x}})$ is the expression of the free surface, and for example, α_{11} means the trace of α_1 in $y = 1$, and the same convention holds for β_{20}. The right hand side of (17) is given by

$$F(\underline{U}) = \begin{cases} -\lambda(1-\rho)e^{-3\beta_{20}} \sin\alpha_{20} - \rho \frac{\partial \alpha_1}{\partial y}_{|y=0} e^{3(\beta_{10}-\beta_{20})}, \\ e^{2\lambda Z - \beta_{11} + \beta_{10} - \beta_{20}} \sin\alpha_{11}, \\ \frac{e^{\beta_{11}}}{2b}(1 - e^{-2(\lambda Z + \beta_{11})})e^{\beta_{10}-\beta_{20}}, \\ \left.\begin{array}{l} \frac{\partial \beta_1}{\partial y} e^{\beta_{10}-\beta_{20}} \\ -\frac{\partial \alpha_1}{\partial y} e^{\beta_{10}-\beta_{20}} \end{array}\right\} y \in (0,1), \\ \left.\begin{array}{l} \frac{\partial \beta_2}{\partial y} \\ -\frac{\partial \alpha_2}{\partial \underline{y}} \end{array}\right\} \underline{y} \in (-\infty, 0). \end{cases} \tag{19}$$

The spaces $\mathbb{H}$ and $\mathbb{D}$ are defined as for problem 1, except that $\mathbb{R}$ is replaced by $\mathbb{R}^3$ and that the boundary condition involving $\alpha_1(y = 1)$ in $\mathbb{D}$ is now $\alpha_{11} = \alpha_1(y = 1)$.

The Galilean invariance of the physical problems induces a reflection symmetry (through the y axis) of both systems in the moving frame. This reflection leads to the reversibility of system (17), i.e. to the existence of a linear symmetry S which anticommutes with the vector field F. This reversibility symmetry is then defined by

$$S\underline{U} = (\beta_{20}, -\alpha_1, \beta_1, -\alpha_2, \beta_2)^t, \text{ for problem 1,}$$
$$S\underline{U} = (\beta_{20}, Z, -\alpha_{11}, -\alpha_1, \beta_1, -\alpha_2, \beta_2)^t, \text{ for problem 2.}$$

The dispersion equation reads (for Re $k > 0$) for problem 1

$$\Delta_1(k) = \rho k \cosh(k) - \sinh(k)[k - \lambda(1 - \rho)] = 0, \tag{20}$$

while for the second problem, this equation reads for Re $k > 0$

$$\begin{aligned} \Delta_2(k) = k \cosh(k)(\rho b k^2 + \lambda - k) \\ - \sinh(k)\left\{(\lambda + bk^2)[\lambda(1-\rho) - k] + \rho k^2\right\} = 0. \end{aligned} \tag{21}$$

Because of reversibility, both expressions should be completed by the symmetric relationship for Re $k < 0$. This means that in Δ_1 and Δ_2, k should be replaced by $(\mathrm{sgn}(k)\mathrm{Re}(k))$. The complex roots of $\Delta_j(k) = 0$ give all complex eigenvalues ik of the linearized operator $D_{\underline{U}}F(0)$ belonging to the upper part of the complex plane. These isolated eigenvalues have a finite multiplicity, and are completed by the symmetric eigenvalues in the lower half plane. They are located in a sectorial region of the complex plane, centered on the real axis, which leads to the finiteness of the number of such eigenvalues in the neighborhood of the imaginary axis (see for example [6]). In addition to this discrete set, the spectrum of the linearized operator contains an *essential spectrum formed by the entire real axis*. This is shown, for example in [6], and this results from the form of the Cauchy-Riemann operator in the infinite layer $(-\infty < \underline{y} < 0)$. Let us give more precisions on the eigenvalues lying on the imaginary axis.

For problem 1, we introduce the parameter $\varepsilon = \rho - \lambda(1-\rho)$. The study of the equation (20) shows that the linear operator $L_\varepsilon = D_{\underline{U}}F(0)$ has the spectrum of Figure 3 . For problem 2, the parameter ε is $1-\lambda(1-\rho)$. The study of the equation $\Delta_2(k) = 0$ for k real and for b large leads to the following conclusion: for $\varepsilon > 0$ small enough, b large enough and $1-\rho = (\alpha/b)^{1/3}$ with $0 < \alpha < 4$, then 0 is the only real solution of the dispersion equation (21). This means that for $\varepsilon < 0$ there is a pair of eigenvalues on the imaginary axis in addition to the 0 eigenvalue, and for $\varepsilon > 0$ this pair disappears (see [3] for the study of the equation $\Delta_1(k) = 0$). Notice that the case treated in [7] is such that $b = 0$, which implies the occurrence of another pair of simple imaginary eigenvalues (given by $k = \pm\lambda$) for all values of ε.

Looking at (18) and (19), we notice the existence of a one parameter family of solutions

$$\underline{U}(x) = u\xi_0, \ u \in \mathbb{R},$$

of the nonlinear system (17). The eigenvector ξ_0 belonging to a zero eigenvalue of the linearized operator (about 0) L_ε reads

$$\xi_0 = (0,0,1,0,0)^t, \text{ for problem 1,}$$
$$\xi_0 = (0,\lambda^{-1},0,0,-1,0,0)^t, \text{ for problem 2.}$$

This family of equilibria corresponds physically to a sliding with a non zero small and uniform velocity of the upper layer over the bottom one.

2.2 Rescaling

Let us introduce the basic rescaling of our systems for $\varepsilon > 0$, hence hiding the pair of imaginary eigenvalues occurring for $\varepsilon < 0$ (the details are only given for problem 2, since the formulation for problem 1 is very similar). We set

$$\varepsilon\underline{x} = x; \ \varepsilon\underline{y} = y, \ y \in (-\infty, 0); \ y = y, \ y \in (0,1)$$
$$\underline{U} = \varepsilon U,$$

and equation (17) now reads

$$\frac{dU}{dx} = \mathcal{L}_\varepsilon U + \mathcal{N}_\varepsilon(U), \tag{22}$$

where for problem 2, $U = (\beta_{20}, Z, \alpha_{11}, \alpha_1, \beta_1, \alpha_2, \beta_2)^t$, and

$$\mathcal{L}_\varepsilon U = \begin{pmatrix} \varepsilon^{-1}\left\{-(1-\varepsilon)\alpha_{10} - \rho \frac{\partial \alpha_1}{\partial y}_{|y=0}\right\} \\ \varepsilon^{-1}\alpha_{11} \\ \varepsilon^{-1} 1/b(\beta_{11} + \lambda Z) \\ \varepsilon^{-1}\frac{\partial \beta_1}{\partial y} \\ -\varepsilon^{-1}\frac{\partial \alpha_1}{\partial y} \\ \frac{\partial \beta_2}{\partial y} \\ -\frac{\partial \alpha_2}{\partial y} \end{pmatrix},$$

and

$$\mathcal{N}_\varepsilon(U) = \varepsilon^{-2}\begin{pmatrix} -(1-\varepsilon)[e^{-3\varepsilon\beta_{20}}\sin(\varepsilon\alpha_{10}) - \varepsilon\alpha_{10}] - \varepsilon\rho\frac{\partial \alpha_1}{\partial y}_{|y=0}[e^{3\varepsilon(\beta_{10}-\beta_{20})} - 1] \\ e^{2\varepsilon\lambda Z - \varepsilon\beta_{11} + \varepsilon\beta_{10} - \varepsilon\beta_{20}}\sin(\varepsilon\alpha_{11}) - \varepsilon\alpha_{11} \\ e^{\varepsilon\beta_{11}}/2b(1 - e^{-2\varepsilon(\lambda Z + \beta_{11})})e^{\varepsilon\beta_{10} - \varepsilon\beta_{20}} - \varepsilon/b(\beta_{11} + \lambda Z) \\ \varepsilon\frac{\partial \beta_1}{\partial y}\left\{e^{\varepsilon(\beta_{10}-\beta_{20})} - 1\right\} \\ -\varepsilon\frac{\partial \alpha_1}{\partial y}\left\{e^{\varepsilon(\beta_{10}-\beta_{20})} - 1\right\} \\ 0 \\ 0 \end{pmatrix}.$$

We observe that (in both problems) the two last components of $\mathcal{N}_\varepsilon(U)$ are zero and that the differentiability in y of the components α_2, β_2 of U is not necessary to define $\mathcal{N}_\varepsilon(U)$. Therefore, the following property holds

$$\mathcal{N}_\varepsilon : \widehat{\mathbb{D}} \to \widetilde{\mathbb{H}},$$

where

$$\begin{aligned}\widehat{\mathbb{D}} = \mathbb{R}^3 \times \left\{\mathcal{C}^1(0,1)\right\}^2 \times \left\{\mathcal{C}^0_1(\mathbb{R}^-)\right\}^2 \\ \cap \left\{\alpha_1(0) = \alpha_2(0), \alpha_{11} = \alpha_1(1), \beta_{20} = \beta_2(0)\right\},\end{aligned}$$

i.e. the functions α_2 and β_2 of the vector U are only continuous when $U \in \widehat{\mathbb{D}}$ and

$$\widetilde{\mathbb{H}} = \mathbb{R}^3 \times \left\{\mathcal{C}^0(0,1)\right\}^2 \times \left\{\mathcal{C}^{\exp}_\varepsilon(\mathbb{R}^-)\right\}^2,$$

where $\mathcal{C}^{\exp}_\varepsilon(\mathbb{R}^-) = \{f \in \mathcal{C}^0(\mathbb{R}^-), \|f\|^{\exp}_\varepsilon < \infty\}$ with the norm for a given $d > 0$

$$\|f\|^{\exp}_\varepsilon = \sup_{y\in\mathbb{R}^-} |f(y)|e^{-dy/2\varepsilon},$$

with similar definitions for problem 1. Actually, $\widetilde{\mathbb{H}}$ could be chosen such that the two last components are 0, but we choose a space for which these components have an exponential decay rate ($\widetilde{\mathbb{H}}$ is then dense in $\mathbb{H}$). Notice that $\mathcal{L}_\varepsilon$ considered in $\mathcal{L}(\widetilde{\mathbb{H}})$ is Fredholm on the real axis. It can be shown that the only eigenvalue is 0, and for any λ real, $\lambda - \mathcal{L}_\varepsilon$ has a closed range, of codimension 1 if $\lambda \neq 0$ and 2 if $\lambda = 0$. Actually, $\widetilde{\mathbb{H}}$ is chosen such that the map $k \mapsto (ik - \mathcal{L}_\varepsilon)^{-1}$ is regular in $\mathcal{L}(\widetilde{\mathbb{H}}, \widehat{\mathbb{D}})$ for k real close to 0 (see section 2.3).

Notice also the existence in both problems of a first integral (hence independent of x), which reads for problem 2

$$h = (1-\varepsilon)\left\{-\frac{1}{2\lambda}e^{-2\lambda\varepsilon Z} - \int_0^1 (e^{-\varepsilon\beta_1}\cos(\varepsilon\alpha_1) - 1)dy\right\} + \frac{1}{2}e^{2\varepsilon\beta_{20}} - \frac{\rho}{2}e^{2\varepsilon\beta_{10}}. \quad (23)$$

This combination of the two Bernoulli first integrals at the free surface and at the interface is well defined in $\mathbb{H}$. The existence of such an integral is not necessary in the study which follows.

2.3 Resolvent operator of $\mathcal{L}_\varepsilon$

This section is devoted to the study of the resolvent operator $(ik - \mathcal{L}_\varepsilon)^{-1}$ for $\varepsilon > 0$ small enough. The explicit formulae can be found for example in [3] and [6]. Here we give the estimates on this resolvent on the imaginary axis (i.e. for k real) near the origin and for $|k|$ large.

For both problems, the resolvent can be written as follows for $\varepsilon|k|$ small enough.

Lemma 2.1 (resolvent operator for small $\boldsymbol{\varepsilon|k|}$) *There exists $\delta > 0$ such that for $k \in \mathbb{R}\backslash\{0\}$, $\varepsilon|k| < \delta$ and $V \in \mathbb{H}$, the resolvent operator is decomposed as follows*

$$(ik - \mathcal{L}_\varepsilon)^{-1}V = \frac{\xi^*_{\varepsilon,k}(V)}{ik\varepsilon\Delta}\xi_0 + \frac{\eta^*_{\varepsilon,k}(V)}{\Delta}\theta_k + \varepsilon S_{\varepsilon,k}(V), \quad (24)$$

with the properties described below in four parts.

The proof of this Lemma and of the properties below lies on the study of the equation $(ik - \mathcal{L}_\varepsilon)U = V$ for a given vector $V \in \mathbb{H}$. The explicit formulae can be found in [3] and [6] and the properties below result easily from these formulae.

Notice that the form (24) of the resolvent is the same as in Hypothesis H1 in [3], where this decomposition is used as a general assumption to describe the spectrum of Figure 3. Actually, the form (24) (or an adaptation of this form, depending in particular on the multiplicity of the 0 eigenvalue) turns out to be a general formulation of the resolvent of the linear operator involved in many water-wave problems, when the bottom layer of fluid is infinitely deep. Therefore, in the rest of the paper, we describe the dynamical system (22) in the most general frame, since we can prove the existence of periodic solutions for other systems having the properties described below for the water waves problems.

Let us now give in four parts the main properties of each term introduced in (24). These properties result from the explicit formulae of the resolvent.

The dispersion equation. *For $k \in \mathbb{R}\backslash\{0\}$, and near 0, the dispersion equation reads*

$$ik\varepsilon\Delta(\varepsilon, k) = 0,$$

where the map $\Delta(\varepsilon, k)$ is even with respect to k and satisfies

$$\Delta(\varepsilon, k) = 1 + a|k| + O(\varepsilon k^2),$$

where $a = 1$ for problem 1, and $a = \lambda^{-1}(\lambda - 1)$ for problem 2.

The root $k = 0$ in the dispersion equation is related to the simple eigenvalue in 0 of $\mathcal{L}_\varepsilon$. We also observe that Δ is not analytic, this is a sign of the fact that 0 lies in the essential spectrum. The evenness comes from the reversibility of the system. For problem 2, the dispersion equation (before scaling) is $\Delta_2(k) = 0$ (see (21)), Δ is obtained by the relation $\Delta(\varepsilon, k) = \Delta_2(\varepsilon k)/(\varepsilon^2 k\lambda)$.

On the splitting of the space and the projection associated with the kernel of $\mathcal{L}_\varepsilon$.

There exists $p_0^* \in \mathbb{H}^*$ such that $p_0^*(SV) = p_0^*(V)$, $p_0^*(\xi_0) = 1$.

There exists $\delta > 0$ such that for $k \in \mathbb{R}\backslash\{0\}$, $\varepsilon|k| \le \delta$ and $V \in \mathbb{H}$, we have

$$p_0^*(ik - \mathcal{L}_\varepsilon)^{-1}V = \frac{\xi_{\varepsilon,k}^*(V)}{ik\Delta(\varepsilon, k)},$$

where, for $k \neq 0$, $\xi_{\varepsilon,k}^* \in \mathbb{H}^*$, $\xi_{\varepsilon,k}^*(SV) = \xi_{\varepsilon,-k}^*(V)$ and we can define the form $\zeta_{\varepsilon,k}^*$ with

$$\begin{aligned} &\xi_{\varepsilon,k}^* = \xi_\varepsilon^* + \zeta_{\varepsilon,k}^*, \quad \xi_\varepsilon^* \in \mathbb{H}^*, \\ &|\zeta_{\varepsilon,k}^*(V)/k| \le c\varepsilon, \text{ for } V \in \widetilde{\mathbb{H}}. \end{aligned}$$

For problem 1, we choose $p_0^*(V) = \beta_{11} = \beta_1(y = 1)$, and for a vector $V = (a, f_1, g_1, f_2, g_2)^t \in \mathbb{H}$

$$\xi_\varepsilon^*(V) = -a + \rho g_{10} - (\rho - \varepsilon) \int_0^1 g_1(\tau)d\tau.$$

For problem 2, we choose $p_0^*(V) = -\beta_{11} = -\beta_1(y = 1)$ and for a vector $V = (a_1, a_2, a_3, f_1, g_1, f_2, g_2)^t \in \mathbb{H}$

$$\xi_\varepsilon^*(V) = a_1 - \rho g_{10} + (1 - \varepsilon)a_2 + (1 - \varepsilon) \int_0^1 g_1(\tau)d\tau.$$

The symmetry property of the linear form $\xi_{\varepsilon,k}^*$ is an easy consequence of the reversibility of (22) and it implies the invariance of ξ_ε^* under S (i.e. $\xi_\varepsilon^*(SV) = \xi_\varepsilon^*(V)$). We can also prove in both problems that $\xi_\varepsilon^*(\xi_0) = \varepsilon$. Therefore, the usual projection on the kernel of the linear operator $\mathcal{L}_\varepsilon$ is $V \mapsto \varepsilon^{-1}\xi_\varepsilon^*(V)$, which is singular when $\varepsilon \to 0^+$, that's why we prefer to use the projection p_0^* which is independent of ε.

Notice that the estimate on $\zeta_{\varepsilon,k}^*$ is not valid in $\mathbb{H}$ but in the smaller space $\widetilde{\mathbb{H}}$. Indeed, the main (non differentiable) term of $\zeta_{\varepsilon,k}^*$ can be written as $\varepsilon|k|\chi_\varepsilon^*$ where

$$\chi_\varepsilon^*(V) = \int_0^1 g_1(\tau)d\tau + \varepsilon^{-1}\int_{-\infty}^0 g_2(\tau)d\tau, \text{ for problem 1,}$$
$$\chi_\varepsilon^*(V) = -a_2 - \int_0^1 g_1(\tau)d\tau + \varepsilon^{-1}\int_{-\infty}^0 g_2(\tau)d\tau, \text{ for problem 2.}$$

We observe that this form cannot be defined in $\mathbb{H}$ because of an insufficient decay rate of the component g_2 in the lower layer of fluid, whereas it is well-defined in $\widetilde{\mathbb{H}}$ (with a uniform bound).

Singularity related to the essential spectrum in 0. *The singular part of* $(ik - \mathcal{L}_\varepsilon)^{-1}V$ *in* $\ker p_0^*$ *reads*

$$\frac{\eta_{\varepsilon,k}^*(V)}{\Delta(\varepsilon,k)}\theta_k,$$

where $k\theta_k$ is bounded in $\mathbb{D}$, $(k\theta_k)_{k=0} = 0$, $S\theta_k = -\theta_{-k}$, $p_0^*(\theta_k) = 0$ and

$$\xi_\varepsilon^*(\theta_k) = i\text{sgn}(k). \tag{25}$$

For $k \neq 0$, $\eta_{\varepsilon,k}^* \in \mathbb{H}^*$, $\eta_{\varepsilon,k}^*(SV) = \eta_{\varepsilon,-k}^*(V)$ and

$$\eta_{\varepsilon,k}^* = \eta_\varepsilon^* + \beta_{\varepsilon,k}^*,\ \eta_\varepsilon^* \in \widetilde{\mathbb{H}}^*,\ \beta_{\varepsilon,0}^* = 0,$$
$$|\beta_{\varepsilon,k}^*(V)/k| \leq c\varepsilon,\ V \in \widetilde{\mathbb{H}}.$$

We obtain the following vector θ_k

$$\theta_k = (-i\text{sgn}(k), y-1, 0, -e^{|k|y}, -i\text{sgn}(k)e^{|k|y})^t, \text{ for problem 1,}$$
$$\theta_k = (i\text{sgn}(k), 0, -(\lambda-1)^{-1}, 1-\lambda(\lambda-1)^{-1}y, 0, e^{|k|y}, i\text{sgn}(k)e^{|k|y})^t, \text{ for problem 2,}$$

from which we directly obtain the relation (25). Since the norm of the function $y \mapsto e^{|k|y}$ in $\mathcal{C}_1^0(\mathbb{R}^-)$ is $1/|k|$ for $|k|$ small, then $k\theta_k$ is bounded in $\mathbb{D}$.

The linear form η_ε^* is defined by

$$\eta_\varepsilon^*(V) = a - \rho g_{10} + \rho\int_0^1 g_1(\tau)d\tau + \int_{-\infty}^0 g_2(\tau)d\tau, \text{ for problem 1,}$$
$$\eta_\varepsilon^*(V) = \lambda^{-1}(1-\lambda)(a_1 - \rho g_{10}) - \rho a_2 - \rho\int_0^1 g_1(\tau)d\tau + \int_{-\infty}^0 g_2(\tau)d\tau, \text{ for problem 2.}$$

We observe that for the same reasons as for χ_ε^*, the linear form η_ε^* cannot be defined in $\mathbb{H}$. Looking at the definitions of ξ_ε^*, η_ε^* and χ_ε^*, we observe that the following relationship holds

$$a\xi_\varepsilon^* = \varepsilon\chi_\varepsilon^* - \eta_\varepsilon^*. \tag{26}$$

It has been seen in [3] that this relation is equivalent to (25).

Regular part of the resolvent near 0 for $k \in \mathbb{R}\backslash\{0\}$. *The regular part of* $(ik-\mathcal{L}_\varepsilon)^{-1}V$ *reads*

$$\varepsilon S_{\varepsilon,k}(V),$$

where $S_{\varepsilon,k} \in \mathcal{L}(\mathbb{H},\mathbb{D})$ *for* $k \neq 0$ *and* $S_{\varepsilon,k}$ *is uniformly bounded with for* $\varepsilon|k| < \delta$ *in* $\mathcal{L}(\widetilde{\mathbb{H}},\widehat{\mathbb{D}})$.

Remark on the range of $\mathcal{L}_\varepsilon$. Looking at (24), we observe that there is a limit when k tends to 0 if $V \in \widetilde{\mathbb{H}}$ and $\xi_\varepsilon^*(V) = \eta_\varepsilon^*(V) = \chi_\varepsilon^*(V) = 0$. These are sufficient conditions on V to be in the range of $\mathcal{L}_\varepsilon$. It results that the range of $\mathcal{L}_\varepsilon$ has codimension 2 since the relation (26) holds and since $\widetilde{\mathbb{H}}$ is dense in $\mathbb{H}$.

We finally give here a property on the resolvent on the imaginary axis for $\varepsilon|k| > \delta/2$. Notice that only the continuity of the resolvent is needed here, whereas we needed the differentiability of $G(\varepsilon,k)$ in [3]. These estimates are obtained as in [6].

Lemma 2.2 (resolvent operator for large $\boldsymbol{\varepsilon|k|}$) *Let* $V \in \mathbb{H}$, *then for* k *real,*

$$(ik-\mathcal{L}_\varepsilon)^{-1}V = G(\varepsilon,k)(V), \tag{27}$$

where $k \mapsto G(\varepsilon,k)$ *is continuous in* $\mathcal{L}(\mathbb{H},\mathbb{D})$ *for* $\varepsilon|k| > \delta/2$ *with the following estimates in* $\mathcal{L}(\mathbb{H})$ *and in* $\mathcal{L}(\widetilde{\mathbb{H}},\widehat{\mathbb{D}})$

$$\|G(\varepsilon,k)\|_{\mathcal{L}(\mathbb{H})} \leq c/|k|, \;\; \|G(\varepsilon,k)\|_{\mathcal{L}(\widetilde{\mathbb{H}},\widehat{\mathbb{D}})} \leq c\varepsilon. \tag{28}$$

2.4 Properties of the non linear term $\mathcal{N}_\varepsilon$

Let us now give the properties of the non linear term. The first Lemma describes the regularity of $\mathcal{N}_\varepsilon$. This Lemma is an easy consequence of the expression of $\mathcal{N}_\varepsilon$.

Lemma 2.3 (properties of the non linear operator $\mathcal{N}_\varepsilon$) *For* $k \geq 3$, *the non linear map* $\mathcal{N}_\varepsilon$ *satisfies*

$$\mathcal{N}_\varepsilon \in \mathcal{C}^k(\widehat{\mathbb{D}},\widetilde{\mathbb{H}}), \;\; D_U\mathcal{N}_\varepsilon(0) = 0,$$
$$D_U^m\mathcal{N}_\varepsilon(0) = O(\varepsilon^{m-2}), \;\; m = 2,3.$$

Moreover, $\mathcal{N}_\varepsilon(\nu\xi_0) = 0$ *for all* $\nu \in \mathbb{R}$ *in a neighborhood of 0.*

Notice that the property $\mathcal{N}_\varepsilon(\nu\xi_0) = 0$ is equivalent to the existence of the one parameter family of stationary solutions $U(x) = \nu\xi_0$ for ν in a neighborhood of 0. It has been seen in [3] that this property is related to the 0 eigenvalue and to the structure of the resolvent near the origin.

As in [3], we now introduce $R_\varepsilon(u,Y) = \varepsilon^{-1}\mathcal{N}_\varepsilon(u\xi_0 + \varepsilon Y)$. This operator is used because we decompose a solution of (22) as $U = u\xi_0 + \varepsilon Y$, $p_0^*(Y) = 0$. Thanks to Lemma 2.3, the operator R_ε has the following properties.

Proposition 2.4 (properties of R_ε) *The operator $R_\varepsilon : \mathbb{R} \times \widehat{\mathbb{D}} \to \widetilde{\mathbb{H}}$ is $\mathcal{C}^k$ and satisfies*

$$R_\varepsilon(u, Y) = uD_\varepsilon Y + \widetilde{R}_\varepsilon(u, Y),$$

where $D_\varepsilon \in \mathcal{L}(\widehat{\mathbb{D}}, \widetilde{\mathbb{H}})$, and

$$\|\widetilde{R}_\varepsilon(u, Y)\|_{\widetilde{\mathbb{H}}} \leq c\varepsilon \|Y\|_{\widehat{\mathbb{D}}}(|u| + \|Y\|_{\widehat{\mathbb{D}}}),$$

for $|u| + \|Y\|_{\widehat{\mathbb{D}}} \leq M$.

For problem 2, a straightforward computation leads to the following operator D_ε

$$D_\varepsilon Y = \left(3\rho \frac{\partial \alpha_1}{\partial y}|_{y=0}, 2\alpha_{11}, -\frac{2\lambda}{b}Z, -\frac{\partial \beta_1}{\partial y}, \frac{\partial \alpha_1}{\partial y}, 0, 0\right)^t.$$

Next proposition gives the link between ξ^*_ε, θ_k and the operator D_ε introduced above. This proposition results from a straightforward computation.

Proposition 2.5 *For both problems, the following property holds*

$$\xi^*_\varepsilon(D_\varepsilon \theta_k) = 2c_0 + \varepsilon\gamma_\varepsilon(k),\ c_0 \neq 0,\ k \mapsto \gamma_\varepsilon(k) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}),$$

with $2c_0 = 3\rho$ for problem 1, and $2c_0 = -3\lambda/(\lambda - 1)$ for problem 2.

Notice that in both problems, the function γ_ε is constant. This is linked with the existence of the Bernoulli first integral (see (23)). Actually, the proof of the existence of periodic solutions of (22) only requires the continuity of the function γ_ε. That's why we prefer to give the most general property in Proposition 2.5, since the study of this article can be generalized to other reversible systems having the spectrum as in Figure 3 .

3 Existence of periodic solutions

3.1 Working system and notations

In this section we come back to the system (22). We fix a real number $T > 0$, and we look for periodic solutions of (22) of period aT (where a is introduced in the definition of the function Δ). Since we want to work with 2π-periodic functions, we perform the following change of variable

$$U(x) = \underline{U}(s),\ s = \frac{2\pi}{aT}x,$$

so that equation (22) now reads

$$\frac{2\pi}{aT}\frac{d\underline{U}}{ds} = \mathcal{L}_\varepsilon \underline{U} + \mathcal{N}_\varepsilon(\underline{U}), \tag{29}$$

As in [3] for the search of homoclinic solutions, we decompose a 2π-periodic solution $\underline{U}$ of (29) as follows

$$\underline{U} = u\xi_0 + \varepsilon Y,\ p_0^*(Y) = 0, \tag{30}$$

where p_0^* is defined in section 2.3, and is such that $p_0^*(\xi_0) = 1$. Thanks to the definition of R_ε (i.e. $R_\varepsilon(u, Y) = \varepsilon^{-1}\mathcal{N}_\varepsilon(u\xi_0 + \varepsilon Y)$), we can write the system for u and Y

$$\frac{2\pi}{aT}\frac{du}{ds} = p_0^*(\mathcal{L}_\varepsilon \varepsilon Y) + p_0^*(\varepsilon R_\varepsilon(u, Y)), \tag{31}$$

$$\frac{2\pi}{aT}\frac{dY}{ds} = \pi(\mathcal{L}_\varepsilon Y) + \pi(R_\varepsilon(u, Y)), \tag{32}$$

where π is defined by $\pi(V) = V - p_0^*(V)\xi_0$ for $V \in \mathbb{H}$.

We look for reversible solutions $\underline{U}$ of (29), i.e. $S\underline{U}(s) = \underline{U}(-s)$. This implies the following for u and Y

$$u \text{ is even, and } SY(s) = Y(-s).$$

To prove the existence of periodic solutions of (29), we first prove that the system (31)-(32) can be reduced, for reversible solutions, to a scalar equation for u. Indeed, if (u, Y) is a reversible solution of (31)-(32), then Y is a function of u for $\varepsilon > 0$ small enough (see Theorem 4.2). The equation satisfied by u is given in Theorem 4.3. This equation is a perturbation of the Benjamin-Ono equation, for which there is a family of periodic solutions (see Theorems 4.4 and 4.7). Therefore, a family of periodic reversible solutions of (29) can be constructed (see Theorems 3.2 and 3.3).

Notations

Before going to the theorems proving the existence of periodic solutions of (22), let us give some definitions.

Let $\mathbb{E}$ be a Banach space and $n \in \mathbb{N}$, we denote by $H^n_\sharp(\mathbb{E})$ the space of 2π-periodic functions u, taking values in $\mathbb{E}$ and such that $\sum_{k\in\mathbb{Z}}(1+|k|^{2n})\|u_k\|^2_{\mathbb{E}} < \infty$, with the usual norm, and where u_k is the k^{th} Fourier coefficient of u. The space of even functions of $H^n_\sharp(\mathbb{E})$ is denoted by $H^n_{\sharp,e}(\mathbb{E})$.

The Hilbert transform for 2π-periodic functions $\mathcal{H}_\sharp$ is defined by the following relations

$$\mathcal{H}_\sharp(\cos) = \sin,\ \mathcal{H}_\sharp(\sin) = -\cos,$$

and $\mathcal{H}_\sharp(const.) = 0$. The Fourier coefficients of $\mathcal{H}_\sharp(f)$ are given by the following relation

$$\{\mathcal{H}_\sharp(f)\}_k = -i\text{sgn}(k)f_k,\ k \in \mathbb{Z}\backslash\{0\}.$$

We now introduce the operators and the vectors linked with the vector θ_k. Indeed, as in [3], the construction of the periodic solutions depends on the vector θ_k.

Therefore, we define the operator $u \mapsto \mathcal{T}_\sharp(u)$ for a function $u \in H^1_\sharp(\mathbb{R})$ by

$$\{\mathcal{T}_\sharp(u)\}_k = -i\frac{2\pi}{T}\varphi_0(2\pi\varepsilon k/aT)ku_k\theta_{\frac{2\pi}{aT}k},\ k \in \mathbb{Z}\backslash\{0\},$$

which is completed by $\{\mathcal{T}_\sharp(u)\}_0 = 0$ (in using $\{k\theta_k\}_{k=0} = 0$). In this formula, we introduced a splitting of the unity $\varphi_0 + \varphi_1 = 1$

$$\varphi_0(\varepsilon k) = \begin{cases} 1,\ \varepsilon|k| < \delta/2 \\ 0,\ \varepsilon|k| > \delta \end{cases},\quad \varphi_1(\varepsilon k) = \begin{cases} 0,\ \varepsilon|k| < \delta/2 \\ 1,\ \varepsilon|k| > \delta \end{cases}.$$

Notice that, thanks to the property $\xi^*_\varepsilon(D_\varepsilon\theta_k) = i\mathrm{sgn}(k)$, we obtain

$$\xi^*_\varepsilon(\mathcal{T}_\sharp(u)) = \frac{2\pi}{T}\mathcal{H}_\sharp\left(\frac{du_0}{ds}\right),$$

where $\{u_0\}_k = \varphi_0(2\pi\varepsilon k/aT)u_k$. Since $k\theta_k$ is bounded in $\mathbb{D}$ in the neighborhood of $k = 0$ the following property holds

Proposition 3.1 *The operator $\mathcal{T}_\sharp$ is bounded from $H^n_\sharp(\mathbb{R})$ into $H^{n-1}_\sharp(\mathbb{D})$, for $n \geq 1$*

$$\mathcal{T}_\sharp \in \mathcal{L}(H^n_\sharp(\mathbb{R}), H^{n-1}_\sharp(\mathbb{D})),$$

and for an even function u, $S\mathcal{T}_\sharp(u)(s) = \mathcal{T}_\sharp(u)(-s)$.

The symmetry property of this proposition results from the symmetry of θ_k : $S\theta_k = -\theta_{-k}$.

Remark. The definition of $\mathcal{T}_\sharp$ shows that if u is a constant function, then $\mathcal{T}_\sharp(u) = 0$.

Water-wave problems. In the second water-wave problem, we compute explicitly the vector $\mathcal{T}_\sharp(u)$ (the result being similar for problem 1). We obtain $\mathcal{T}_\sharp(u) = (\beta_{20}, 0, \alpha_{11}, \alpha_1, 0, \alpha_2, \beta_2)^t$ where

$$\alpha_1(y) = -\frac{2\pi}{T}\left(1 - \frac{\lambda}{\lambda - 1}y\right)u_0',$$
$$\alpha_2(y) = -\frac{1}{T}\int_{-\pi}^{\pi}\frac{u_0'(s-\tau)(1 - e^{\frac{4\pi}{aT}y})}{1 - 2e^{\frac{2\pi}{aT}y}\cos(\tau) + e^{\frac{4\pi}{aT}y}}d\tau,$$
$$\beta_2(y) = \frac{1}{T}\int_{-\pi}^{\pi}\frac{2u_0'(s-\tau)\sin(\tau)e^{\frac{2\pi}{aT}y}}{1 - 2e^{\frac{2\pi}{aT}y}\cos(\tau) + e^{\frac{4\pi}{aT}y}}d\tau,$$

we check in particular that

$$\beta_{20} = \frac{2\pi}{T}\mathcal{H}_\sharp(u_0').$$

3.2 Main Theorems

The following theorem shows the existence of a family of periodic solutions of (29). These solutions are close to stationary solutions belonging to the family of equilibria $u\xi_0$, $u \in \mathbb{R}$.

Theorem 3.2 *There exists $\varepsilon_0 > 0$, $\kappa_1 > \kappa_0 > 0$ and $A_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the equation (29) has a family of reversible periodic solutions $\underline{\mathcal{U}}^A_{\kappa,\varepsilon} \in H^2_\sharp(\mathbb{R})\xi_0 \oplus \{H^1_\sharp(\widehat{\mathbb{D}}) \cap ker\, p_0^*\}$, parametrized by A ($|A| < A_0$) and $\kappa \in [\kappa_0, \kappa_1]$. The parameter T is given by*

$$T(\varepsilon, \kappa, A^2) = \frac{2\pi}{\kappa(1-\mu)},\ \mu(\varepsilon, \kappa, A^2) = \frac{1}{2}(\kappa^{-1}ac_0)^2A^2 + O(\varepsilon).$$

These solutions read

$$\underline{\mathcal{U}}^A_{\kappa,\varepsilon}(s) = u^0_{\kappa,\varepsilon}\xi_0 + A\ \underline{\mathcal{U}}_{\kappa,\varepsilon}(s) + \widetilde{\underline{\mathcal{U}}}^A_{\kappa,\varepsilon}(s),$$

with

$$u^0_{\kappa,\varepsilon} = -\frac{\kappa+1}{2ac_0} + O(\varepsilon) \text{ is a constant,}$$

and $\underline{\mathcal{U}}_{\kappa,\varepsilon}(s) = \cos(s)\xi_0 + O(\varepsilon)$, $\widetilde{\underline{\mathcal{U}}}^A_{\kappa,\varepsilon} = O(A^2)$.

Remark. This theorem shows that the stationary solutions $u^0_{\kappa,\varepsilon}\xi_0$, belonging to the family of equilibria $U = u\xi_0$, $u \in \mathbb{R}$ are limits of periodic solutions of amplitude $O(A)$ tending towards 0. Actually, we can prove that the linearized operator around these stationary solutions $u^0_{\kappa,\varepsilon}\xi_0$, i.e. $\mathcal{L}_\varepsilon + D_U\mathcal{N}_\varepsilon(u^0_{\kappa,\varepsilon}\xi_0)$, has two imaginary eigenvalues $\pm i\sigma_{\kappa,\varepsilon} = \pm i2\pi/(aT(\varepsilon,\kappa,0)) = \pm i\kappa/a + O(\varepsilon)$. Therefore, the equilibria $u\xi_0$ are elliptic for u large enough.

We note that for $\varepsilon > 0$ small enough and κ close to 1, there is also a family of equilibria of the form $u\xi_0$, with $u \simeq (\kappa-1)/(2ac_0)$, hence close to 0 (see section 4.2 and the resolution of (71)). These equilibria are exactly the ones of [3] to which the family of homoclinic solutions with a polynomial decay rate are connected.

The solutions are searched in $H^2_\sharp(\mathbb{R})\xi_0 \oplus \{H^1_\sharp(\widehat{\mathbb{D}}) \cap \ker p_0^*\}$. This space has been chosen in order to work in an algebra : if $u \in H^2_\sharp(\mathbb{R})$ and if $Y \in H^1_\sharp(\widehat{\mathbb{D}})$ then the term $R_\varepsilon(u, Y)$ of proposition 2.4 is in $H^1_\sharp(\widetilde{\mathbb{H}})$. We need more regularity for the function u, since Y is computed with $\mathcal{T}_\sharp(u)$, which is in $H^1_\sharp(\mathbb{D})$ for $u \in H^2_\sharp(\mathbb{R})$.

The preceding theorem shows the existence of periodic solutions close to the equilibria $u^0_{\kappa,\varepsilon}\xi_0$. In next theorem, we prove that there are also "large periodic solutions".

Theorem 3.3 *There exists a sub-set $\mathcal{P}$ of $(2\pi, +\infty)$, which differs from the interval $(2\pi, +\infty)$ by a discrete set with no accumulation point, for which the following result holds : for all compact set $\mathcal{K} \subset \mathcal{P}$, there exist $\varepsilon_0 > 0$, $\kappa_1 > \kappa_0 > 0$ such that* *the equation (29) has a family of periodic solutions $\underline{\mathcal{V}}^p_{\kappa,\varepsilon} \in H^2_\sharp(\mathbb{R})\xi_0 \oplus \{H^1_\sharp(\widehat{\mathbb{D}}) \cap \ker p_0^*\}$, parametrized by $p \in \mathcal{K}$ and $\kappa \in (\kappa_0, \kappa_1)$. The parameter T is given by $T = p/\kappa$, and $\underline{\mathcal{V}}^p_{\kappa,\varepsilon}$ satisfies*

$$\underline{\mathcal{V}}^p_{\kappa,\varepsilon}(s) = v^p_{\kappa,\varepsilon}\xi_0 + \widetilde{\underline{\mathcal{V}}}^p_{\kappa,\varepsilon}(s),$$

with $v^p_{\kappa,\varepsilon}(s) = u_{\kappa,p}(s) + O(\varepsilon)$, where

$$u_{\kappa,p} = -\frac{\kappa}{ac_0}v_p + \frac{\kappa-1}{2ac_0}, \tag{33}$$

and where v_p is defined in (4). Finally, $\widetilde{\underline{\mathcal{V}}}^p_{\kappa,\varepsilon} = O(\varepsilon)$.

We recover the solutions of Theorem 3.2 when p tends toward 2π in $\mathcal{V}^p_{\kappa,\varepsilon}$. Notice that, in this Theorem, we cannot reach the homoclinic solutions found in [3] (which correspond to solitary waves) since we cannot reach the value $p = +\infty$.

Remark. The results of Theorems 3.2 and 3.3 are given in the general spatial dynamics framework, since the study presented here can be applied to other reversible systems having the spectrum as in Figure 3 and having the properties presented in sections 2.3 and 2.4. In next paragraph, we write these results for both water-wave problems.

Periodic waves. The periodic solutions of Theorem 3.2 and 3.3 correspond to periodic waves for both problems. In problem 1, we can compute the expression of the interface $Z_{I,1}$ (using $2ac_0 = 3\rho$)

$$Z_{I,1}(x) \sim -\varepsilon\left(\frac{\kappa+1}{3\rho} + A\cos\left(\frac{2\pi}{T}\varepsilon x\right)\right), \tag{34}$$

where T is close to $2\pi/\kappa$.

For problem 2, we obtain the following expression of the free surface Z (using $ac_0 = -3/2$)

$$Z(x) \sim \frac{\varepsilon}{\lambda}\left(\frac{\kappa+1}{3} + A\cos\left(\frac{2\pi}{aT}\varepsilon x\right)\right), \tag{35}$$

and the expression of the interface $Z_{I,2}$

$$Z_{I,2}(x) \sim -\varepsilon\frac{\lambda-1}{\lambda}\left(\frac{\kappa+1}{3} + A\cos\left(\frac{2\pi}{aT}\varepsilon x\right)\right). \tag{36}$$

This result shows the existence of periodic travelling waves superposed to the uniform translation of the upper layer of fluid. Notice in problem 2 the opposition of phases between the oscillations of the free surface and the ones of the interface.

As for the previous Theorem, the periodic solutions $\mathcal{V}^p_{\kappa,\varepsilon}$ of (22) lead to periodic waves for problem 1 and 2, of period (in the unscaled variables) $ap/(\kappa\varepsilon)$ (see (10) for the expression of the interface for problem 1, and (11)-(12) for the expression of the interface and free surface for problem 2). Notice that we recover the expressions (34), (35) and (36) (i.e. when the amplitude A of the periodic waves is small) by taking p close to 2π in (10), (11) and (12).

The rest of this paper is devoted to the proof of the Theorems 3.2 and 3.3. In section 4.1 the system (31)-(32) is reduced to an equation for u, which is a perturbation of the Benjamin-Ono equation. Then, in section 4.2, we prove that this equation has a family of periodic solutions, from which we obtain the families of solutions given in Theorems 3.2 and 3.3.

4 Proofs of Theorems 3.2 and 3.3

4.1 Reduction to the Benjamin-Ono equation

The aim of this section is to reduce the system (31)-(32) to a non-local scalar equation for the function u. To obtain this equation, we first solve in section 4.1.1 the linear non-homogeneous system obtained from (31)-(32). Then, in section 4.1.2 we prove that the function Y is determined by u for $\varepsilon > 0$ close to 0 and that u is a solution of a non-local scalar equation, which is a perturbation of the Benjamin-Ono equation.

Notice that this strategy of reduction is similar to the one performed in [3], except that we now work with periodic functions instead of functions which decay towards 0. This implies in particular the presence of the constant c appearing in the Benjamin-Ono equation (64) of Theorem 4.3.

4.1.1 Linear lemma

We study the non-homogeneous linear system obtained from (31)-(32) where we replaced $R_\varepsilon(u, Y)$ by a given anti-reversible function $R \in H^1_\sharp(\widetilde{\mathbb{H}})$ (i.e. R satisfies $SR(s) = -R(-s)$). This system reads

$$\frac{2\pi}{aT}\frac{du}{ds} = p_0^*(\mathcal{L}_\varepsilon \varepsilon Y) + p_0^*(\varepsilon R), \tag{37}$$

$$\frac{2\pi}{aT}\frac{dY}{ds} = \pi(\mathcal{L}_\varepsilon Y) + \pi(R). \tag{38}$$

The following lemma gives the solution u and Y of (37)-(38) as functions of R.

Lemma 4.1 *Let $R \in H^1_\sharp(\widetilde{\mathbb{H}})$ be an anti-reversible function. Let $(u, Y) \in H^2_\sharp(\mathbb{R}) \times H^1_\sharp(\widehat{\mathbb{D}})$ be a solution of (37)-(38), then*

$$Y = \mathcal{T}_\sharp(u) + \mathcal{T}_{\sharp,\varepsilon}(R), \tag{39}$$

$$\frac{2\pi}{aT}\frac{d}{ds}\left\{u + \frac{2\pi}{T}\mathcal{H}_\sharp(u') + \mathcal{C}_{\sharp,\varepsilon}(R)\right\} = \xi_\varepsilon^*(R), \tag{40}$$

with

$$\|\mathcal{T}_{\sharp,\varepsilon}(R)\|_{H^1_\sharp(\widehat{\mathbb{D}})} \leq c\varepsilon \|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}, \tag{41}$$

$$\|\mathcal{C}_{\sharp,\varepsilon}(R)\|_{H^1_\sharp(\mathbb{R})} \leq c\varepsilon \|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}. \tag{42}$$

Moreover, $S\mathcal{T}_{\sharp,\varepsilon}(R)(s) = \mathcal{T}_{\sharp,\varepsilon}(R)(-s)$ *and* $\mathcal{C}_{\sharp,\varepsilon}(R)$ *is even.*

Notice that the formulation (39)-(40) is a weaker formulation of (37)-(38) since $\mathcal{L}_\varepsilon Y$ is not in $\mathbb{H}$ for $Y \in \widehat{\mathbb{D}}$. Actually, this formulation is weak in the sense that we solve the linear system for its Fourier coefficients, and it can be understood in the distribution sense. In our water-wave problems, we recover regular functions thanks to properties of the Cauchy-Riemann equations in the half plane.

The idea of the proof is to write Fourier coefficients of the system (37)-(38) in order to use the properties of the resolvent. The equation (39) is obtained in observing that the coefficients of Y and of u are linked thanks to the operator $\mathcal{T}_\sharp$. The difficulty is then to prove that the remaining term, i.e. $\mathcal{T}_{\sharp,\varepsilon}(R)$ is small. Finally, we use $\xi^*_\varepsilon(\mathcal{T}_\sharp(u)) = (2\pi/T)\mathcal{H}(\partial_x u_0)$ to prove (40).

Proof of Lemma 4.1. Let us write the Fourier coefficients of the system (37)-(38). We then obtain the following equation for $k \in \mathbb{Z}$

$$(i\tilde{k} - \mathcal{L}_\varepsilon)(u_k \xi_0 + \varepsilon Y_k) = \varepsilon R_k, \tag{43}$$

where u_k, Y_k and R_k are the k-th Fourier coefficients of u, Y and R, and $\tilde{k} = 2\pi k/(aT)$. Let us define $u_{k,0} = u_k \varphi_0(\varepsilon\tilde{k})$ and $u_{k,1} = u_k \varphi_1(\varepsilon\tilde{k})$ and similarly $Y_{k,0}$ et $Y_{k,1}$.

First step : $\boldsymbol{\varepsilon|\tilde{k}| < \delta}$. We first consider the following equation

$$(i\tilde{k} - \mathcal{L}_\varepsilon)(u_{k,0}\xi_0 + \varepsilon Y_{k,0}) = \varphi_0(\varepsilon\tilde{k})\varepsilon R_k.$$

The property of the resolvent for $\varepsilon|k|$ small, allows us to solve this equation and to express $u_{k,0}$ and $Y_{k,0}$ in function of R_k

$$u_{k,0} = \frac{\varphi_0 \xi^*_\varepsilon(R_k)}{i\tilde{k}\Delta} + \{S^{(0)}_{\sharp,u}(R)\}_k, \tag{44}$$

$$Y_{k,0} = -a\frac{\xi^*_\varepsilon(R_k)}{\Delta}\varphi_0\theta_{\tilde{k}} + \{S^{(0)}_{\sharp,Y}(R)\}_k, \tag{45}$$

where we defined for $k \neq 0$

$$\{S^{(0)}_{\sharp,u}(R)\}_k = \varphi_0 \frac{\zeta^*_{\varepsilon,\tilde{k}}(R_k)}{i\tilde{k}\Delta}, \tag{46}$$

and

$$\{S^{(0)}_{\sharp,Y}(R)\}_k = \varphi_0 \frac{\varepsilon\chi^*_\varepsilon(R_k)}{\Delta}\theta_{\tilde{k}} + \varphi_0 \frac{\beta^*_{\varepsilon,\tilde{k}}(R_k)}{\Delta}\theta_{\tilde{k}} + \varphi_0 \varepsilon S_{\varepsilon,\tilde{k}}(R_k). \tag{47}$$

Notice that the function $k \mapsto \{S^{(0)}_{\sharp,u}(R)\}_k$ is even. Indeed, $\zeta^*_{\varepsilon,k}(SV) = \zeta_{\varepsilon,-k}(V)$ and the functions $k \mapsto \Delta$ and $k \mapsto \varphi_0$ are even. Moreover, $SR_k = -R_{-k}$ since R is anti-symmetric. This explains the evenness of $k \mapsto \{S^{(0)}_{\sharp,u}(R)\}_k$. Because of similar symmetry relations for $\eta^*_{\varepsilon,k}$ and $S_{\varepsilon,k}$ and thanks to the relation $S\theta_k = -\theta_{-k}$, we obtain $S\{S^{(0)}_{\sharp,Y}(R)\}_k = \{S^{(0)}_{\sharp,Y}(R)\}_{-k}$.

Note also that the relations (44) and (45) are well-defined for $k = 0$. Indeed, since $R_0 \in \widetilde{\mathbb{H}}$ and satisfies $SR_0 = -R_0$, then $\xi^*_\varepsilon(R_0) = \eta^*_\varepsilon(R_0) = 0$. It results that R_0 is in the range of $\mathcal{L}_\varepsilon$, and this allows to define $Y_{0,0}$ and $u_{0,0}$.

Estimates on $\boldsymbol{S^{(0)}_{\sharp,u}(R)}$ and $\boldsymbol{S^{(0)}_{\sharp,Y}(R)}$. We now estimate $S^{(0)}_{\sharp,u}(R)$ and $S^{(0)}_{\sharp,Y}(R)$. Using the following estimate

$$\frac{1+k^2}{\Delta(\varepsilon,\tilde{k})^2} \le c\frac{1+k^2}{(1+a|\tilde{k}|)^2} \le c\max\left(1, \frac{T^2}{(2\pi)^2}\right), \tag{48}$$

and the fact that the function $k \mapsto \zeta^*_{\varepsilon,k}/k$ is uniformly bounded by ε in $\widetilde{\mathbb{H}}^*$ for $\varepsilon|k| < \delta$, we obtain

$$(1+k^2)|\{S^{(0)}_{\sharp,u}(R)\}_k|^2 \le c\varepsilon^2\|R_k\|^2_{\widetilde{\mathbb{H}}}.$$

We deduce that $S^{(0)}_{\sharp,u}(R) \in H^2_\sharp(\mathbb{R})$ with

$$\|S^{(0)}_{\sharp,u}(R)\|_{H^2_\sharp(\mathbb{R})} \le c\varepsilon\|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}. \tag{49}$$

We use the same technics to estimate $S^{(0)}_{\sharp,Y}(R)$. The first term of $S^{(0)}_{\sharp,Y}(R)$, denoted by $S^{(0)}_{\sharp,Y,1}(R)$, can be written as

$$\left\{S^{(0)}_{\sharp,Y,1}(R)\right\}_k = \varphi_0\frac{\varepsilon\chi^*_\varepsilon(R_k)}{\Delta}\theta_{\tilde{k}} = ik\varepsilon\varphi_0\frac{\left\{A(R)\right\}_k}{\Delta}\theta_{\tilde{k}},$$

where

$$A(R) = \int_0^x \chi^*_\varepsilon(R)ds.$$

The linear form χ^*_ε is invariant under S and R is anti-reversible. Therefore, the function $\chi^*_\varepsilon(R)$ is odd. This proves that the function $A(R)$ is in $H^2_\sharp(\mathbb{R})$ with $\|A(R)\|_{H^2_\sharp(\mathbb{R})} \le c\|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}$. We deduce the following estimate thanks to Proposition 3.1

$$\|S^{(0)}_{\sharp,Y,1}(R)\|_{H^1_\sharp(\widehat{\mathbb{D}})} \le c\varepsilon\|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}. \tag{50}$$

The estimate on the second term of (47), denoted $S^{(0)}_{\sharp,Y,2}(R)$, is obtained as for the first term. We have indeed

$$\left\{S^{(0)}_{\sharp,Y,2}(R)\right\}_k = \varphi_0\frac{\beta^*_{\varepsilon,\tilde{k}}(R_k)}{\Delta}\theta_{\tilde{k}} = i\tilde{k}\varphi_0\{B(R)\}_k\theta_{\tilde{k}},$$

where

$$\{B(R)\}_k = \varphi_0 \frac{\beta^*_{\varepsilon,\tilde{k}}(R_k)}{i\tilde{k}\Delta}.$$

Since $\beta^*_{\varepsilon,k}/k$ is uniformly bounded by ε in $\widetilde{\mathbb{H}}$ for $\varepsilon|k| < \delta$ and thanks to (48), $B(R) \in H^2_\sharp(\mathbb{R})$ with $\|B(R)\|_{H^2_\sharp(\mathbb{R})} \leq c\varepsilon\|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}$. It results from Proposition 3.1 that $S^{(0)}_{\sharp,Y,2}(R) \in H^1(\widehat{\mathbb{D}})$ with

$$\|S^{(0)}_{\sharp,Y,2}(R)\|_{H^1_\sharp(\widehat{\mathbb{D}})} \leq c\varepsilon\|R\|_{H^1_\sharp(\mathbb{H})}. \tag{51}$$

The last term of (47) can be estimated as follows

$$\|\varepsilon\varphi_0 S_{\varepsilon,\tilde{k}}(R_k)\|_{\widehat{\mathbb{D}}} \leq c\varepsilon\|R_k\|_{\mathbb{H}}, \tag{52}$$

since the function $k \mapsto S_{\varepsilon,\tilde{k}}$ is uniformly bounded in $\mathcal{L}(\widetilde{\mathbb{H}},\widehat{\mathbb{D}})$ for $\varepsilon|\tilde{k}| < \delta$. We deduce from (50), (51) and (52) the following result : $S^{(0)}_{\sharp,Y}(R) \in H^1_\sharp(\widehat{\mathbb{D}})$ with

$$\|S^{(0)}_{\sharp,Y}(R)\|_{H^1_\sharp(\widehat{\mathbb{D}})} \leq c\varepsilon\|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}. \tag{53}$$

Relation between $\boldsymbol{u_{k,0}}$ and $\boldsymbol{Y_{k,0}}$. From (44)-(45), we obtain the following relation between $Y_{k,0}$ and $u_{k,0}$

$$Y_{k,0} = -i\frac{2\pi}{T}ku_{k,0}\theta_{\tilde{k}} + \{\mathcal{T}^0_{\sharp,\varepsilon}(R)\}_k, \tag{54}$$

with

$$\{\mathcal{T}^0_{\sharp,\varepsilon}(R)\}_k = S^{(0)}_{\sharp,Y}(R_k) + i\frac{2\pi}{T}kS^{(0)}_{\sharp,u}(R_k)\theta_{\tilde{k}}. \tag{55}$$

An estimate on $\mathcal{T}^0_{\sharp,\varepsilon}(R)$ in $H^1_\sharp(\widehat{\mathbb{D}})$ is obtained in gathering (49), (53) and in using Proposition 3.1

$$\|\mathcal{T}^0_{\sharp,\varepsilon}(R)\|_{H^1_\sharp(\widehat{\mathbb{D}})} \leq c\varepsilon\|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}. \tag{56}$$

Moreover, thanks to the symmetry relation of $S^{(0)}_{\sharp,Y}(R)$ and thanks to evenness $k \mapsto \{S^{(0)}_{\sharp,u}(R)\}_k$, we remark that the expression $\mathcal{T}^0_{\sharp,\varepsilon}(R)$ is anti-reversible

$$S\mathcal{T}^0_{\sharp,\varepsilon}(R)(x) = \mathcal{T}^0_{\sharp,\varepsilon}(R)(-x). \tag{57}$$

Second step : $\boldsymbol{\varepsilon|\tilde{k}| > \delta}$. We now study the equation for $u_{k,1}$ and $Y_{k,1}$, i.e.

$$(i\tilde{k} - \mathcal{L}_\varepsilon)(u_{k,1}\xi_0 + \varepsilon Y_{k,1}) = \varphi_1(\varepsilon\tilde{k})\varepsilon R_k.$$

Thanks to Lemma 2.2, $u_{k,1}$ and $Y_{k,1}$ are given by the following relations

$$u_{k,1} = \varepsilon\varphi_1(\varepsilon\tilde{k})p^*_0(G(\varepsilon,\tilde{k})(R_k)),\ Y_{k,1} = \varphi_1(\varepsilon\tilde{k})\pi(G(\varepsilon,\tilde{k})(R_k)).$$

Since $p_0^*(SV) = p_0^*(V)$, $S\pi = \pi S$ and $SG(\varepsilon,k)(V) = -G(\varepsilon,-k)(SV)$ for all V in $\mathbb{H}$, the following symmetry relations hold

$$u_{k,1} = u_{-k,1}, \ \ SY_{k,1} = Y_{-k,1}. \tag{58}$$

To estimate $Y_{k,1}$, we use $\|G(\varepsilon,k)\|_{\mathcal{L}(\widetilde{\mathbb{H}},\widehat{\mathbb{D}})} \le c\varepsilon$ which shows that $Y_1 \in H^1_\sharp(\widehat{\mathbb{D}})$ with

$$\|Y_1\|_{H^1_\sharp(\widehat{\mathbb{D}})} \le c\varepsilon\|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}. \tag{59}$$

To estimate u_1, we use $\|G(\varepsilon,k)\|_{\mathcal{L}(\mathbb{H})} \le c/|k|$, then

$$(1+k^2)|u_{k,1}|^2 \le c\varepsilon^2\frac{1+k^2}{|\tilde{k}|^2}\|R_k\|^2_{\widetilde{\mathbb{H}}}.$$

It results that $u_1 \in H^2_\sharp(\mathbb{R})$ with

$$\|u_1\|_{H^2_\sharp(\mathbb{R})} \le c\varepsilon\|R\|_{H^1_\sharp(\widetilde{\mathbb{H}})}. \tag{60}$$

Equation for $\boldsymbol{Y}$. Adding $Y_{k,1}$ on both sides of (54), we obtain

$$Y_k = -\varphi_0 i\frac{2\pi}{T}ku_k\theta_{\tilde{k}} + \{\mathcal{T}_{\sharp,\varepsilon}(R)\}_k\,, \tag{61}$$

where we defined the operator $\mathcal{T}_{\sharp,\varepsilon}(R)$ by

$$\{\mathcal{T}_{\sharp,\varepsilon}(R)\}_k = \{\mathcal{T}^0_{\sharp,\varepsilon}(R)\}_k + Y_{k,1}.$$

The equation (61) corresponds to the equation (39) of Lemma 4.1. The estimate (41) is obtained in gathering (56), (59) and (60) together with Proposition 3.1. Finally, the symmetry relation of Lemma 4.1 results from (57) and (58).

Notice that, applying ξ^*_ε on equation (61), we obtain

$$\xi^*_\varepsilon(Y) = \frac{2\pi}{T}\mathcal{H}_\sharp(u_0') + \xi^*_\varepsilon(\mathcal{T}_{\sharp,\varepsilon}(R)). \tag{62}$$

Equation for $\boldsymbol{u}$. To end the proof of Lemma 4.1, it remains to find the equation giving u in function of R. We apply the linear form ξ^*_ε on the system (37)-(38), then using $\xi^*_\varepsilon(\mathcal{L}_\varepsilon U) = 0$ we obtain

$$\xi^*_\varepsilon\left(\frac{2\pi}{aT}\frac{d}{ds}\{u\xi_0 + \varepsilon Y\}\right) = \varepsilon\xi^*_\varepsilon(R).$$

Finally, with the relation (62) and $\xi^*_\varepsilon(\xi_0) = \varepsilon$ we obtain the equation (40) with $\mathcal{C}_{\sharp,\varepsilon}(R) = \xi^*_\varepsilon(\mathcal{T}_{\sharp,\varepsilon}(R)) - 2\pi/T\mathcal{H}_\sharp(u_1')$. This concludes the proof of Lemma 4.1. $\square$

4.1.2 Reduction

In the previous section, we have solved the linear system. We now replace the function R in Lemma 4.1 by the non linear term $R_\varepsilon(u,Y)$. We observe that if $u \in H^2_\sharp(\mathbb{R})$ is even and if $Y \in H^1_\sharp(\widehat{\mathbb{D}})$ is reversible then $R_\varepsilon(u,Y) \in H^1_\sharp(\widetilde{\mathbb{H}})$ is anti-reversible and satisfies the assumptions of Lemma 4.1. The equation (39), in which R is replaced by $R_\varepsilon(u,Y)$, shows that the function $Y \in H^1_\sharp(\widehat{\mathbb{D}})$ is a solution of

$$Y = \mathcal{T}_\sharp(u) + \mathcal{T}_{\sharp,\varepsilon}(R_\varepsilon(u,Y)). \tag{63}$$

This equation can be solved with respect to Y thanks to the implicit function theorem.

Theorem 4.2 *For all $M > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and for all function $u \in H^2_{\sharp,e}(\mathbb{R})$ (i.e. $u \in H^2_\sharp(\mathbb{R})$ is even) such that $\|u\|_{H^2_\sharp(\mathbb{R})} \leq M$, a reversible solution $(u,Y) \in H^2_\sharp(\mathbb{R}) \times H^1_\sharp(\widehat{\mathbb{D}})$ of (31)-(32) must satisfy*

$$\begin{aligned} Y &= \mathcal{Y}_{\sharp,\varepsilon}(u), \\ &= \mathcal{T}_\sharp(u) + O(\varepsilon), \end{aligned}$$

where the function $\mathcal{Y}_{\sharp,\varepsilon}$ is smooth $H^2_{\sharp,e}(\mathbb{R}) \mapsto H^1_\sharp(\widehat{\mathbb{D}})$.

Proof of Theorem 4.2. We solve equation (63) with the implicit function theorem. Here u is fixed and is such that $\|u\|_{H^2_\sharp(\mathbb{R})} \leq M$, and we use the estimate (41). Actually, we need to adapt this theorem since $\mathcal{T}_{\sharp,\varepsilon}(R_\varepsilon(u,Y))$ is defined only for $\varepsilon > 0$. Thus, we replace $\mathcal{T}_{\sharp,\varepsilon}(R_\varepsilon(u,Y))$ by

$$\mathcal{T}_{\sharp,\varepsilon}(R_\varepsilon(u,Y)) - (1-\mu\varepsilon^{-1})\mathcal{T}_{\sharp,\varepsilon}(R_\varepsilon(u,\mathcal{T}_\sharp(u))).$$

For $\mu = 0$ we have the solution $Y = \mathcal{T}_\sharp(u)$, and equation (63) corresponds to $\mu = \varepsilon$. Applying the implicit function theorem for μ close to 0 and Y near $\mathcal{T}_\sharp(u)$, we obtain the required result in making $\mu = \varepsilon$, which lies in the domain of existence of the solution. $\square$

Remark. In the case when u is constant, then $\mathcal{Y}_{\sharp,\varepsilon}(u) = 0$. Indeed, we know that $\mathcal{T}_\sharp(u) = 0$ and the uniqueness of the solution of (63) leads to $Y = 0$.

Theorem 4.2 reduces the system (31)-(32) to one equation for the scalar function u. This equation is obtained by replacing R by $R_\varepsilon(u, \mathcal{Y}_{\sharp,\varepsilon}(u))$ in (40). Finally, the proposition 2.5 allows us to rewrite this equation as a perturbation of the Benjamin-Ono equation. Indeed, we prove that the main term of $\xi^*_\varepsilon(R_\varepsilon(u,\mathcal{Y}_{\sharp,\varepsilon}(u)))$ is a local quadratic term. We finally prove that the right hand side of (40) is at main order a local term, which is $-c_0(2\pi/T)\partial_x(u^2)$. This leads to the following result

Theorem 4.3 *Let $(u, Y) \in H^2_\sharp(\mathbb{R}) \times H^1_\sharp(\widehat{\mathbb{D}})$ be a reversible solution of (31)-(32). Then for all $0 < \varepsilon < \varepsilon_0$, the function u satisfies*

$$u + \frac{2\pi}{T}\mathcal{H}_\sharp(u') + ac_0 u^2 = c + \mathcal{B}_{\sharp,\varepsilon}(u), \tag{64}$$

where $\mathcal{B}_{\sharp,\varepsilon} : H^2_{\sharp,e}(\mathbb{R}) \mapsto H^1_{\sharp,e}(\mathbb{R})$ is smooth and $O(\varepsilon)$ and c is a constant of integration.

Proof of Theorem 4.3. We use the following notation in the rest of the paper : $\mathcal{R}_\varepsilon(u) = R_\varepsilon(u, \mathcal{Y}_{\sharp,\varepsilon}(u))$. If $u \in H^2_{\sharp,e}(\mathbb{R})$ then $\mathcal{R}_\varepsilon(u)$ satisfies the assumptions of Lemma 4.1. Therefore, we can replace R by $\mathcal{R}_\varepsilon(u)$ in equation (40)

$$\frac{2\pi}{aT}\frac{d}{ds}\left\{u + \frac{2\pi}{T}\mathcal{H}_\sharp(u') + \mathcal{C}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))\right\} = \xi^*_\varepsilon(\mathcal{R}_\varepsilon(u)), \tag{65}$$

Thanks to Proposition 2.5 we first prove that the main term of $\xi^*_\varepsilon(\mathcal{R}_\varepsilon(u))$ is a local quadratic term : $-c_0 2\pi/T(u^2)'$. The term $\xi^*_\varepsilon(\mathcal{R}_\varepsilon(u))$ appearing in (65) can be computed in using the expansion of R_ε given in Proposition 2.4

$$\xi^*_\varepsilon(\mathcal{R}_\varepsilon(u)) = u\xi^*_\varepsilon(D_\varepsilon\mathcal{Y}_{\sharp,\varepsilon}(u)) + \xi^*_\varepsilon(\widetilde{R}_\varepsilon(u, \mathcal{Y}_{\sharp,\varepsilon}(u))), \tag{66}$$

We now use the relation (61), which gives the Fourier coefficients of $\mathcal{Y}_{\sharp,\varepsilon}(u)$ in function of u_k and the Fourier coefficients of $\mathcal{R}_\varepsilon(u)$. This allows to compute the term $\xi^*_\varepsilon(D_\varepsilon\mathcal{Y}_{\sharp,\varepsilon}(u))$.

$$\begin{aligned}
\left\{\xi^*_\varepsilon(D_\varepsilon\mathcal{Y}_{\sharp,\varepsilon}(u))\right\}_k &= -i\frac{2\pi}{T}ku_{k,0}\xi^*_\varepsilon(D_\varepsilon\theta_{\tilde{k}}) + \xi^*_\varepsilon(D_\varepsilon\left\{\mathcal{T}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))\right\}_k) \\
&= -i\frac{2\pi}{T}ku_k\varphi_0(2c_0 + \varepsilon\gamma_\varepsilon(\tilde{k})) + \xi^*_\varepsilon(D_\varepsilon\left\{\mathcal{T}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))\right\}_k) \\
&= -2c_0\frac{2\pi}{T}\{u'\}_k + 2c_0\varphi_1 iku_k - i\frac{2\pi}{T}ku_k\varphi_0\varepsilon\gamma_\varepsilon(\tilde{k}) \\
&\quad + \xi^*_\varepsilon(D_\varepsilon\left\{\mathcal{T}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))\right\}_k)
\end{aligned} \tag{67}$$

notice that $\xi^*_\varepsilon(D_\varepsilon\theta_k) = 2c_0 + \varepsilon\gamma_\varepsilon(k)$ has been used. We deduce from (67) that

$$\xi^*_\varepsilon(D_\varepsilon\mathcal{Y}_{\sharp,\varepsilon}(u)) = -2c_0\frac{2\pi}{T}\frac{du}{ds} + J^0_{\sharp,\varepsilon}(u), \tag{68}$$

where

$$\left\{J^0_{\sharp,\varepsilon}(u)\right\}_k = 2c_0\varphi_1 iku_k - i\frac{2\pi}{T}ku_k\varphi_0\varepsilon\gamma_\varepsilon(\tilde{k}) + \xi^*_\varepsilon(D_\varepsilon\left\{\mathcal{T}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))\right\}_k). \tag{69}$$

We know that the function γ_ε is even, that $\mathcal{T}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))$ is reversible and that the operator D_ε anti-commutes with S. It results that $J^0_{\sharp,\varepsilon}(u)$ is odd when u is even. We know that $u_1 \in H^2_\sharp(\mathbb{R})$ with the estimate (60). Since the function γ_ε is bounded, we deduce the following estimate on $J^0_{\sharp,\varepsilon}(u)$ (we also use (41))

$$\|J^0_{\sharp,\varepsilon}(u)\|_{H^1_\sharp(\mathbb{R})} \le c\varepsilon\|\mathcal{R}_\varepsilon(u)\|_{H^1_\sharp(\widetilde{\mathbb{H}})}.$$

Replacing (68) in (66) then in (65), the equation (65) becomes

$$\frac{2\pi}{aT}\frac{d}{ds}\big\{u+\frac{2\pi}{T}\mathcal{H}_\sharp(u')+\mathcal{C}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))\big\} = -c_0\frac{2\pi}{T}\frac{du^2}{ds}+uJ^0_{\sharp,\varepsilon}(u)+\xi^*_\varepsilon(\widetilde{R}_\varepsilon)$$

which can be written as follows

$$\frac{d}{ds}\big\{u+\frac{2\pi}{T}\mathcal{H}_\sharp(u')+ac_0u^2\big\} = B_{\sharp,\varepsilon}(u), \tag{70}$$

where

$$B_{\sharp,\varepsilon}(u) = -\frac{d}{ds}\big\{\mathcal{C}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))\big\}+\frac{aT}{2\pi}\Big(uJ^0_{\sharp,\varepsilon}(u)+\xi^*_\varepsilon(\widetilde{R}_\varepsilon(u,\mathcal{Y}_{\sharp,\varepsilon}(u)))\Big).$$

Since $\mathcal{T}_{\sharp,\varepsilon}(\mathcal{R}_\varepsilon(u))$ is reversible and since $\widetilde{R}_\varepsilon(u,\mathcal{Y}_{\sharp,\varepsilon}(u))$ is anti-reversible, the function $B_{\sharp,\varepsilon}(u)$ is odd. After an integration in (70), we obtain (64), the periodic function $\mathcal{B}_{\sharp,\varepsilon}(u)$ being the primitive of $B_{\sharp,\varepsilon}(u)$, with zero mean value. □

4.2 Resolution of the perturbed Benjamin-Ono equation

In the previous section, the system (31)-(32) is reduced to equation (64) for the scalar function u, where T and c are parameters. The limit equation, i.e. the equation obtained when $\mathcal{B}_{\sharp,\varepsilon}(u)\equiv 0$, can be solved in the case when $1+4ac_0c>0$ (see [2]). More precisely, the family of functions $u_{\kappa,p}$ parametrized by $\kappa>0$ and $p>2\pi$, defined in (33), is a solution of the limit equation when $T=p/\kappa$ and $c=(\kappa^2-1)/4ac_0$. Therefore, we look for solutions of (64) where we introduce the new parameters $T=p/\kappa$ and $c=(\kappa^2-1)/4ac_0$, with $p>2\pi$ and $\kappa>0$, i.e. we are now interested in the equation

$$u+\kappa\frac{2\pi}{p}\mathcal{H}_\sharp(u')+ac_0u^2 = \frac{\kappa^2-1}{4ac_0}+\mathcal{B}_{\sharp,\varepsilon}(u). \tag{71}$$

The rest of section 4.2 is devoted to the construction of solutions of (71). In section 4.2.1, we find solutions of (71) with p close to 2π. In section 4.2.2, we find solutions of (71) for almost all the values of $p>2\pi$. We also show how these solutions lead to the periodic solutions of Theorems 3.2 and 3.3.

Notice that the resolution of the perturbed Benjamin-Ono equation for functions u tending towards 0 at infinity is easily performed in [3] thanks to a result of [1] on the invertibility of the linearized Benjamin-Ono operator. In next sections, we study this resolution in the periodic case. In particular, we need to study the invertibility of the linearized Benjamin-Ono operator around a given solution (see the definition of L_p in (89)).

4.2.1 Solutions of (71) with p close to 2π

The critical value of p in $u_{\kappa,p}$ is 2π. For that particular value, we obtain the constant function $u_{\kappa,2\pi}$ (denoted by u_κ in the following)

$$u_\kappa(s) = -\frac{\kappa+1}{2ac_0}. \tag{72}$$

Therefore, we look for solutions u of (71) close to u_κ, such that p is close to 2π. The following theorem proves the existence of a family of solutions $u^A_{\kappa,\varepsilon} = u_\kappa + A\cos + O(\varepsilon + A^2)$ parametrized by A, close to 0, and κ, which lies in an interval $[\kappa_0, \kappa_1]$ with $\kappa_0 > 0$. The values of κ are limited since we must avoid the critical value $\kappa = 0$ and because u_κ must be bounded.

Theorem 4.4 *There exists $\varepsilon_0 > 0$, $\kappa_1 > \kappa_0 > 0$ and $A_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ equation (71) has a family of periodic solutions $u^A_{\kappa,\varepsilon}$ parametrized by A ($|A| < A_0$) and $\kappa \in [\kappa_0, \kappa_1]$, and satisfying*

$$u^A_{\kappa,\varepsilon}(x) = u^0_{\kappa,\varepsilon} + A\cos(x) + \widetilde{u}^A_{\kappa,\varepsilon}(x), \tag{73}$$

where $u^0_{\kappa,\varepsilon} = u_\kappa + O(\varepsilon)$ is a constant, and the function $\widetilde{u}^A_{\kappa,\varepsilon} \in H^2_{\sharp,e}(\mathbb{R})$ is of order $O(\varepsilon A + A^2)$. The parameter $p = p(\varepsilon, \kappa, A^2)$ is given by

$$p = \frac{2\pi}{1-\mu}, \; \mu = \frac{1}{2}(\kappa^{-1}ac_0)^2 A^2 + O(\varepsilon).$$

Remark. The constant u_κ is a solution of (71) when $\mathcal{B}_{\sharp,\varepsilon}(u) = 0$. This equation admits another constant solution $\tilde{u}_\kappa$ given by

$$\tilde{u}_\kappa = \frac{-1+\kappa}{2ac_0}.$$

We can prove (same proof as the proof of Theorem 4.4) that equation (71) admits a family of solutions $\tilde{u}^0_{\kappa,\varepsilon} = \tilde{u}_\kappa + O(\varepsilon)$, parametrized by $\kappa \in (\kappa_0, \kappa_1)$. This proves the existence of a family of stationary solutions of (22) which read $U(x) = \tilde{u}^0_{\kappa,\varepsilon}\xi_0$ and which correspond, for $\kappa \sim 1$ (i.e. $\tilde{u}_\kappa$ close to 0), to the "hyperbolic" equilibria of system (22) which we encounter in [3] (these equilibria are not properly hyperbolic, because there is no real positive or negative eigenvalues). The Theorem 4.4 shows that the equilibria u_κ are "elliptic".

Before proving Theorem 4.4, let us give the end of proof of Theorem 3.2.

End of proof of Theorem 3.2. The periodic solutions of Theorems 3.2 are obtained from the solutions $u^A_{\kappa,\varepsilon}$ of (71) and thanks to Theorem 4.2 which allows to construct the solutions of (29). Indeed, if u is a solution of (71) then $U = u\xi_0 + \varepsilon\mathcal{Y}_{\sharp,\varepsilon}(u)$ is a solution of (29). Therefore, we obtain a family of 2π-periodic solutions of (29) : $\underline{\mathcal{U}}^A_{\kappa,\varepsilon} = u^A_{\kappa,\varepsilon}\xi_0 + \varepsilon\mathcal{Y}_{\sharp,\varepsilon}(u^A_{\kappa,\varepsilon})$, by using the Theorem 4.2 with $u = u^A_{\kappa,\varepsilon}$. Hence Theorem 3.2 is proved.

Proof of Theorem 4.4. We look for even solutions of (71) close to u_κ. Therefore, we introduce the function $w \in H^2_{\sharp,e}(\mathbb{R})$, perturbation of the constant function u_κ, and the parameter μ by

$$u = u_\kappa + w, \; \frac{2\pi}{p} = 1 - \mu. \tag{74}$$

The equation satisfied by w is obtained in replacing (74) in (71) and can be written as

$$Lw = N_{\varepsilon,\kappa,\mu}(w). \tag{75}$$

where operators $L \in \mathcal{L}(H^2_{\sharp,e}(\mathbb{R}), H^1_{\sharp,e}(\mathbb{R}))$ and $N_{\varepsilon,\kappa,\mu} : H^2_{\sharp,e}(\mathbb{R}) \mapsto H^1_{\sharp,e}(\mathbb{R})$ are defined by

$$\begin{aligned} Lw &= -w + \mathcal{H}_\sharp(w'), \\ N_{\varepsilon,\kappa,\mu}(w) &= \kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa + w) - \kappa^{-1}ac_0w^2 + \mu\mathcal{H}_\sharp(w'). \end{aligned} \tag{76}$$

For $\|w\|_{H^2_\sharp(\mathbb{R})} \leq \delta$, the following estimate on $N_{\varepsilon,\kappa,\mu}(w)$ holds, for $\kappa \in [\kappa_0, \kappa_1]$

$$\|N_{\varepsilon,\kappa,\mu}(w)\|_{H^1_\sharp(\mathbb{R})} \leq c\left(\varepsilon + \|w\|^2_{H^2_\sharp(\mathbb{R})} + |\mu|\right).$$

Since $(Lw)_k = (-1+|k|)w_k$, the kernel of L in $H^2_{\sharp,e}(\mathbb{R})$ is spanned by the function cos. Hence, we introduce the projection $P(f) = P_0(f)\cos$ on the kernel of L, where for a real valued function f

$$P_0(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\cos(\tau)d\tau,$$

and the projection on the orthogonal of the kernel : $Q = \mathbb{I} - P$. We can now define a pseudo-inverse $\widetilde{L}^{-1} \in \mathcal{L}(H^1_{\sharp,e}(\mathbb{R}), H^2_{\sharp,e}(\mathbb{R}))$ of L. If $N \in H^1_{\sharp,e}(\mathbb{R})$ satisfies $P(N) = 0$ then

$$\widetilde{L}^{-1}(N)(s) = \sum_{k\in\mathbb{Z}\setminus\{-1,1\}} \frac{1}{-1+|k|} N_k e^{iks}.$$

Stationary solutions of (75). We first look for stationary solutions w of (75) close to 0. The equation for w (independent of μ) reads

$$-w = \kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa + w) - \kappa^{-1}ac_0w^2. \tag{77}$$

We solve this equation thanks to the implicit function theorem : for ε positive close to 0, there exists a constant $\mathcal{W}_{K,\varepsilon}$ solution of (77) which satisfies

$$\mathcal{W}_{\kappa,\varepsilon} = -\kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa) + O(\varepsilon^2).$$

It results that (71) admits a family stationary solutions $u^0_{\kappa,\varepsilon} = u_\kappa + \mathcal{W}_{\kappa,\varepsilon}$, where u_κ is defined by (72) and p is arbitrary.

Non constant solutions of (75). To find the non constant even solutions w of (75), we use the Liapunov-Schmidt method. We decompose w as follows

$$w(s) = v(s) + A\cos(s),\ \ P(v) = 0,\ \ v \text{ even}.$$

The equation (75) is then equivalent to the system

$$v = \widetilde{L}^{-1} Q N_{\varepsilon,\kappa,\mu}(v(s) + A\cos(s)), \tag{78}$$

$$0 = \int_{-\pi}^{\pi} N_{\varepsilon,\kappa,\mu}(v(\tau) + A\cos(\tau))\cos(\tau)d\tau. \tag{79}$$

The equation (78) can be solved with respect to v thanks to the implicit function theorem, for ε and μ close to 0, there exists a function $v = \mathcal{V}_{\varepsilon,\kappa}(\mu, A)$ solution of (78). When $A = 0$, then the function $\mathcal{V}_{\varepsilon,\kappa}(\mu, 0)$ is constant and corresponds to the constant $\mathcal{W}_{\kappa,\varepsilon}$ obtained in the previous paragraph. It results from the uniqueness of $\mathcal{V}_{\varepsilon,\kappa}(\mu, A)$, from the invariance of (78) under the change of variables $s \to s + \pi$, $A \to -A$ and from the invariance $s \to -s$, the following properties

$$\mathcal{V}_{\varepsilon,\kappa}(\mu, A)(s + \pi) = \mathcal{V}_{\varepsilon,\kappa}(\mu, -A)(s), \tag{80}$$

$$\mathcal{V}_{\varepsilon,\kappa}(\mu, A)(-s) = \mathcal{V}_{\varepsilon,\kappa}(\mu, A)(s). \tag{81}$$

We compute the main terms of $\mathcal{V}_{\varepsilon,\kappa}(\mu, A)$ thanks to (78) and thanks to the definition of $N_{\varepsilon,\kappa,\mu}$ in (76)

$$\begin{aligned}
\mathcal{V}_{\varepsilon,\kappa}(\mu, A) &= \widetilde{L}^{-1} Q N_{\varepsilon,\kappa,\mu}(A\cos + \mathcal{V}_{\varepsilon,\kappa}(\mu, A)),\\
&= \widetilde{L}^{-1} Q \left\{\mu A \mathcal{H}(\cos') + \kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa) - \kappa^{-1} a c_0 A^2 \cos^2\right\}\\
&\quad + \kappa^{-1}\widetilde{L}^{-1} Q \left\{D_u \mathcal{B}_{\sharp,\varepsilon}(u_\kappa).A\cos\right\} + \text{h.o.t.},
\end{aligned}$$

Observe that $\mathcal{H}(\cos') = \cos$ and $Q(\cos) = 0$. Now using $\widetilde{L}^{-1}(const) = -const$, $Q(const) = const$ and $\widetilde{L}^{-1} Q(\cos^2) = -\sin^2$, we obtain

$$\mathcal{V}_{\varepsilon,\kappa}(\mu, A) = -\kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa) + \kappa^{-1} a c_0 A^2 \sin^2 + \text{h.o.t.} = O(\varepsilon) + O(A^2). \tag{82}$$

Let now replace v by $\mathcal{V}_{\varepsilon,\kappa}(\mu, A)$ in equation (79), which now reads

$$0 = \frac{1}{\pi}\int_{-\pi}^{\pi} N_{\varepsilon,\kappa,\mu}(\mathcal{V}_{\varepsilon,\kappa}(\mu, A)(\tau) + A\cos(\tau))\cos(\tau)d\tau, \tag{83}$$

and can be written as

$$h_{\varepsilon,\kappa}(\mu, A) = 0.$$

We know that $h_{\varepsilon,\kappa}(\mu, 0) = 0$ since $A = 0$ corresponds to the family of stationary solutions. The property (80) shows that $h_{\varepsilon,\kappa}$ is odd with respect to A. Hence, we can write

$$h_{\varepsilon,\kappa}(\mu, A) = A g_{\varepsilon,\kappa}(\mu, A^2).$$

In order to write the main terms of $g_{\varepsilon,\kappa}(\mu, A^2)$, we must compute the beginning of the expansion of $N_{\varepsilon,\kappa,\mu}(A\cos(\tau) + \mathcal{V}_{\varepsilon,\kappa}(\mu, A)(\tau))$

$$\begin{aligned}
N_{\varepsilon,\kappa,\mu}(A\cos(\tau) + \mathcal{V}_{\varepsilon,\kappa}(\tau)) &= \mu A\cos(\tau) + \mu\mathcal{H}_\sharp(\mathcal{V}'_{\varepsilon,\kappa}) - \kappa^{-1} a c_0 (A\cos(\tau) + \mathcal{V}_{\varepsilon,\kappa})^2\\
&\quad + \kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa + A\cos + \mathcal{V}_{\varepsilon,\kappa}),\\
&= \mu A\cos(\tau) - \kappa^{-1} a c_0 A^2\cos^2(\tau) - 2\kappa^{-1} a c_0 A\cos(\tau)\mathcal{V}_{\varepsilon,\kappa}\\
&\quad + \kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa) + \text{h.o.t.}
\end{aligned} \tag{84}$$

We use this expansion to express the main terms of $h_{\varepsilon,\kappa}$, in replacing (84) in (83)

$$h_{\varepsilon,\kappa}(\mu, A) = \mu A - 2\kappa^{-1} a c_0 A \frac{1}{\pi}\int_{-\pi}^{\pi} \mathcal{V}_{\varepsilon,\kappa}(\mu, A)(\tau)\cos^2(\tau) d\tau + \text{h.o.t.} \tag{85}$$

From (82) we can compute the main terms of the following integral

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} \mathcal{V}_{\varepsilon,\kappa}(\tau)\cos^2(\tau)d\tau &= -\kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa) + \kappa^{-1} a c_0 A^2 \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(\tau)\cos^2(\tau)d\tau + \text{h.o.t.}\\ &= -\kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_\kappa) + \frac{1}{4}\kappa^{-1} a c_0 A^2 + \text{h.o.t.} \end{aligned}\tag{86}$$

Replacing (86) in (85), we obtain

$$g_{\varepsilon,\kappa}(\mu, A^2) = \mu - \frac{1}{2}(\kappa^{-1} a c_0)^2 A^2 + 2\kappa^{-2} a c_0 \mathcal{B}_{\sharp,\varepsilon}(u_\kappa) + \text{h.o.t.}$$

Therefore, we can solve $g_{\varepsilon,\kappa}(\mu, A^2) = 0$ with respect to μ and we obtain

$$\mu(\varepsilon, \kappa, A^2) = \frac{1}{2}(\kappa^{-1} a c_0)^2 A^2 + O(\varepsilon). \tag{87}$$

In conclusion, the family $u^A_{\kappa,\varepsilon}(s) = u^0_{\kappa,\varepsilon} + A\cos(s) + \widetilde{u}^A_{\kappa,\varepsilon}(s)$ where $\widetilde{u}^A_{\kappa,\varepsilon} = \mathcal{V}_{\varepsilon,\kappa}(\mu, A) - \mathcal{V}_{\varepsilon,\kappa}(0,0)$ is a solution of (71) with $2\pi/p = 1-\mu$ and μ is given by (87). This concludes the proof of the theorem 4.4. □

4.2.2 Resolution of (71) with $p > 2\pi$

In this section, we solve equation (71) for an open set a values of p. We look for even solutions u approximated by $u_{\kappa,p}$. Therefore, we set

$$u = u_{\kappa,p} + w,\ \ w \in H^2_{\sharp,e}(\mathbb{R}).$$

The function w is a solution of the following equation

$$L_p w = \kappa^{-1}\mathcal{B}_{\sharp,\varepsilon}(u_{\kappa,p} + w) - \kappa^{-1} a c_0 w^2, \tag{88}$$

where we denote by L_p the operator $H^2_\sharp(\mathbb{R}) \mapsto H^1_\sharp(\mathbb{R})$ defined by

$$L_p w = w + \frac{2\pi}{p}\mathcal{H}_\sharp(w') - 2v_p w, \tag{89}$$

where v_p is defined in (4). We intend to solve equation (88) in w with the implicit function theorem. Therefore, we need to study the invertibility of L_p. In the previous section, we have seen that L_p is not invertible for $p = 2\pi$, and that its kernel in $H^2_\sharp(\mathbb{R})$ is spanned by the functions cosine and sine. We prove here that for p close to 2π and $p > 2\pi$ then $L_p \in \mathcal{L}(H^2_{\sharp,e}(\mathbb{R}), H^1_{\sharp,e}(\mathbb{R}))$ is invertible (whereas the kernel of L_p in a space including odd functions is non void, $\partial_s v_p$ being always in the kernel).

Lemma 4.5 *There exists $\tau_0 > 0$ such that for all $0 < \tau < \tau_0$, if*

$$p = 2\pi\left(1 - \frac{1}{2}\tau^2\right)^{-1},$$

then the operator $L_p \in \mathcal{L}(H^2_{\sharp,e}(\mathbb{R}), H^1_{\sharp,e}(\mathbb{R}))$ is invertible.

Proof of Lemma 4.5. To prove this lemma we solve the equation

$$L_p w = f,\ f \in H^1_{\sharp,e}(\mathbb{R}). \tag{90}$$

We define the real number $\tau \geq 0$ by

$$\frac{2\pi}{p} = 1 - \frac{1}{2}\tau^2,$$

so that $v_0 = 1 + \tau + O(\tau^3)$ and we obtain the following expansion of v_p

$$v_p = 1 + \tau \cos + O(\tau^2).$$

The operator L_p can also be expanded as follows

$$L_p w = Lw + \tau L^{(1)}_\tau w + \tau^2 L^{(2)}_\tau w, \tag{91}$$

where L is defined by $Lw = -w + \mathcal{H}_\sharp(w')$ and $L^{(1)}_\tau$ and $L^{(2)}_\tau$ are given by the following relations

$$\begin{aligned} L^{(1)}_\tau w &= -2w\cos, \\ L^{(2)}_\tau w &= -\frac{1}{2}\mathcal{H}_\sharp(w') - 2w(\cos^2 - 1) + O(\tau)w. \end{aligned}$$

Since the kernel of L in $H^2_{\sharp,e}(\mathbb{R})$ is spanned by the function cosine, we decompose w as follows

$$w = w_0 \cos + v,\ v \in H^2_{\sharp,p}(\mathbb{R}),\ P_0(v) = 0,\ w_0 \in \mathbb{R}.$$

Replacing w in (90) and using the expansion of L_p given in (91), we obtain

$$L(v) - 2\tau w_0 \cos^2 - 2\tau v \cos + \tau^2 L^{(2)}_\tau(w_0 \cos + v) = f. \tag{92}$$

To solve this equation, we use the Liapounov-Schmidt method. We project (92) on the range of L by using the operator $Q(v) = v - P_0(v)\cos$ and on the kernel of L by using P_0. We also need the following relations $P_0(\cos) = 1$, $P_0(\cos^2) = 0$. It results that the equation (92) is equivalent to the following system

$$L(v) - 2\tau w_0 \cos^2 - 2\tau Q(v\cos) + \tau^2 Q L^{(2)}_\tau(w_0 \cos + v) = Q(f), \tag{93}$$

$$-2\tau P_0(v\cos) + \tau^2 P_0 L^{(2)}_\tau(w_0 \cos + v) = P_0(f), \tag{94}$$

Using the pseudo-inverse $\widetilde{L}^{-1}$ of L, equation (93) can be written as

$$\mathcal{L}_\tau(v) = \widetilde{L}^{-1}\left\{Q(f) + 2\tau w_0 \cos^2 - \tau^2 w_0 Q L_\tau^{(2)}(\cos)\right\}, \tag{95}$$

where $\mathcal{L}_\tau \in \mathcal{L}(H^2_{\sharp,e}(\mathbb{R}))$ is defined by

$$\begin{aligned}\mathcal{L}_\tau(v) &= \widetilde{L}^{-1} Q L_p(v) = v - 2\tau \widetilde{L}^{-1} Q(v\cos) + \tau^2 \widetilde{L}^{-1} Q L_\tau^{(2)}(v),\\ &= (\mathbb{I} + O(\tau))v.\end{aligned}$$

In particular, for $0 \le \tau < \tau_0$, $\mathcal{L}_\tau$ is invertible and its inverse reads

$$\mathcal{L}_\tau^{-1} v = v + 2\tau \widetilde{L}^{-1} Q(v\cos) + O(\tau^2)v.$$

Using $\widetilde{L}^{-1}(\cos^2) = -\sin^2$, the equation (95) becomes

$$\mathcal{L}_\tau(v) = \widetilde{L}^{-1}\left\{Q(f)\right\} - 2\tau w_0 \sin^2 - \tau^2 w_0 \widetilde{L}^{-1}\left\{Q L_\tau^{(2)}(\cos)\right\}. \tag{96}$$

Now using $\mathcal{L}_\tau^{-1}(\sin^2) = \sin^2 + O(\tau)$, we deduce that, for $0 \le \tau < \tau_0$, $v = v_\tau(w_0, f)$ is given by

$$\begin{aligned}v_\tau(w_0, f) &= \mathcal{L}_\tau^{-1}\widetilde{L}^{-1}\left\{Q(f)\right\} - 2\tau w_0 \mathcal{L}_\tau^{-1}(\sin^2) - \tau^2 w_0 \mathcal{L}_\tau^{-1}\widetilde{L}^{-1}\left\{Q L_\tau^{(2)}(\cos)\right\},\\ &= \mathcal{L}_\tau^{-1}\widetilde{L}^{-1}\left\{Q(f)\right\} - 2\tau w_0 \sin^2 + O(\tau^2) w_0.\end{aligned} \tag{97}$$

We now replace v by $v_\tau(w_0, f)$ given by (97) in (94)

$$-2\tau P_0(v_\tau(w_0, f)\cos) + \tau^2 P_0 L_\tau^{(2)}(w_0 \cos + v_\tau(w_0, f)) = P_0(f). \tag{98}$$

We compute $P_0(v_\tau(w_0, f)\cos)$ by using the relation (97)

$$P_0(v_\tau(w_0, f)\cos) = P_0(\mathcal{L}_\tau^{-1}\widetilde{L}^{-1}\left\{Q(f)\right\}\cos) - \frac{1}{2}\tau w_0 + O(\tau^2) w_0, \tag{99}$$

since $P_0(\sin^2\cos) = 1/4$. Now, using (99) in (98), we find

$$\tau^2 w_0 + O(\tau^3) w_0 = P_0(f) + O(\tau) f. \tag{100}$$

For $0 < \tau < \tau_0$, we can solve (100) and we obtain w_0 in function of f, then $v_\tau(w_0, f)$ and finally $w \in H^2_{\sharp,e}(\mathbb{R})$ solution of (90). □

Remark. The operator L_p is not invertible in $\mathcal{L}(H^2_\sharp(\mathbb{R}), H^1_\sharp(\mathbb{R}))$ since the odd function $\partial_s v_p$ is in the kernel of L_p for the values of $p \ge 2\pi$.

The following lemma proves that L_p is invertible for a given open set of values of p.

Lemma 4.6 *There exists a set $\mathcal{P} \subset (2\pi, +\infty)$ which differs from the interval $(2\pi, +\infty)$ by a discrete set without point of accumulation, such that for all $p \in \mathcal{P}$, the operator $L_p \in \mathcal{L}(H^2_{\sharp,e}(\mathbb{R}), H^1_{\sharp,e}(\mathbb{R}))$ is invertible.*

Proof of Lemma 4.6. We define the operator $T_p \in \mathcal{L}(H^2_{\sharp,e}(\mathbb{R}))$ by the relation

$$T_p(w) = 2\left(1 + \frac{2\pi}{p}\mathcal{H}_\sharp(\partial_x)\right)^{-1}(v_p w),\ w \in H^2_{\sharp,e}(\mathbb{R}). \tag{101}$$

The family T_p (Re $p > 2\pi$) is an analytic family of compact operators and satisfies

$$1 - T_p = \left(1 + \frac{2\pi}{p}\mathcal{H}_\sharp(\partial_x)\right)^{-1} L_p. \tag{102}$$

Thanks to Lemma 4.5, we know that $1 - T_p$ is invertible for $p > 2\pi$ close to 2π, then the function $p \mapsto (1 - T_p)^{-1}$ is meromorphic in the neighborhood of $(2\pi, +\infty)$ (see Theorem 1 in [9]). It results that there is a discrete set without point of accumulation of values of $p > 2\pi$ for which $1 - T_p$ is not invertible. □

We now solve (88) with respect to w thanks to the implicit function theorem and obtain the following result:

Theorem 4.7 *Let $p \in \mathcal{P}$, then there exist $\varepsilon_0 > 0$ (which depends on p), $\kappa_1 > \kappa_0 > 0$, such that for all $0 < \varepsilon < \varepsilon_0$ and $\kappa \in (\kappa_0, \kappa_1)$, equation (71) admits an even solution $v^p_{\kappa,\varepsilon} \in H^2_{\sharp,e}(\mathbb{R})$ which satisfies*

$$v^p_{\kappa,\varepsilon}(s) = u_{\kappa,p}(s) + O(\varepsilon). \tag{103}$$

Let us now give the end of proof of Theorem 3.3.

End of proof of Theorem 3.3. The periodic solutions of Theorems 3.3 are obtained from the solutions $v^p_{\kappa,\varepsilon}$ of (71) and thanks to Theorem 4.2.

Theorem 4.7 proves the existence of a family of solutions $v^p_{\kappa,\varepsilon}$ of (64) with $p \in \mathcal{P}$ (i.e. for almost all the values of $p \in (2\pi, +\infty)$) and $c = (\kappa^2 - 1)/(2ac_0)$. Thanks to Theorem 4.2, we obtain the family of periodic solutions $\underline{\mathcal{V}}^p_{\kappa,\varepsilon}$ of Theorem 3.3

$$\underline{\mathcal{V}}^p_{\kappa,\varepsilon} = v^p_{\kappa,\varepsilon}\xi_0 + \varepsilon\mathcal{Y}_{\sharp,\varepsilon}(v^p_{\kappa,\varepsilon}).$$

Notice that in this theorem, the values of p lies in a compact set $\mathcal{K} \subset \mathcal{P}$ in order to have an ε_0 which does not depend on p. Hence Theorem 3.3 is proved.

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